Examples relating to no-arbitrage concepts in discrete time

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Conférence en l'honneur de Nicole El Karoui, Mai 2019 à Paris.

Outline

- Introduction and problem description
- Recalling basic notions and results from the literature
- Tools for investigating no-arbitrage notions weaker than classical NA
- Examples

Problem description

- After the seminal work by Delbaen and Schachermayer (DS) leading to NFLVR, the interest arose in finding weaker notions of no-arbitrage that still allow to solve basic problems such as pricing and hedging and portfolio optimization.
- A breakthrough came with the work of Karatzas and Kardaras (KK) (2007) leading to the notions of NUPBR or, equivalently, NA1 (see also the notion of BK in Kabanov (K)('97)) that correspond to minimal conditions to solve meaningfully portfolio optimization problems (see also Fernholz and Karatzas (2009)). In parallel there was the benchmark approach by Platen et al).

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A basic question arises: are there significant examples for market models in between NFLVR and NUPBR (an issue related to the existence of strict local martingales in the context of bubbles)

Problem description

In a continuous-time context it is
not easy to find significant and realistic models. Examples appear in:

- Ruf,R. '14 "A Systematic Approach to Constructing Market Models With Arbitrage"
- Aksamit,Choulli, Deng, Jeanblanc '14 "Arbitrages in a progressive enlargement setting"
- Fontana, Jeanblanc, Song '14 "On arbitrages arising with honest times"
- Chau, R.,Tankov '18 "Arbitrage and utility maximization in market models with an insider"
- In Mancin,R.'14 a jump-diffusion model with restrictions on the investment strategies beyond the natural constraints (non-negative portfolio values): the density process of a candidate martingale measure turns out to be a supermartingale that is not even a local martingale.

Problem description

Consequently there may be interest in investigating the possible impact of restrictions beyond the natural constraints in solving of the basic problems also with weaker no-arbitrage concepts (for multifactor models one may consider also the impact of the support and of the dependence structure).

→ **Intuition:** even if there may be arbitrage, the additional restrictions may not allow for arbitrarily scalable arbitrage.

Problem description

Here an attempt to study this problem in a discrete time setting

- It is commonly known that in discrete time and under natural constraints the various possible no-arbitrage concepts are equivalent.
- One more reason to consider supplementary constraints.

Problem description

We base ourselves on Korn and Schäl (KS)(2009) "The numeraire portfolio in discrete time: existence, related concepts and applications".

- They consider as weaker no-arbitrage notion that of "No unbounded increasing profit" (NUIP) that can be considered as discrete time counterpart of NUPBR (NA1).

→ In fact, they show that, if NUIP fails, then the expected utility maximization problem has either infinitely many solutions or no solution at all.

- Differently from this approach, Baptiste, Carassus, Lepinette (2019) introduce the notions of "absence of (weak) immediate profit" (A(W)IP) for discrete time models and discuss their relation with other NA notions; they study super-replicating pricing by convex duality methods and present numerical experiments.

Problem description

Ours is work in progress: rather than complete results, here basically only examples to illustrate situations that may arise.

- For simplicity of presentation, only the single-period case; extensions to the multiperiod case are not difficult, in particular if the underlying price process forms a Markov process (for log-utility the optimal strategies are anyway of the myopic type).

$\longrightarrow$ We start by recalling some notions from (KS).

Model

- A market with one non-risky asset S^0 and $d \geq 1$ risky assets S^i, $(i = 1, \cdots, d)$.

  $\rightarrow$ Assume $S^0 \equiv 1$ $\quad \rightarrow \quad$ prices S^i already discounted.

- Dynamics: $\quad S_1^i = S_0^i(1 + R^i)\,; \quad S_0^i = 1$
- Self-financing investment strategy $\pi = (\pi^1, \cdots, \pi^d)$ *(ratios invested)*
- (Discounted) Portfolio value
  $V_1^\pi = 1 + \sum_{i=1}^d \pi^i(S_1^i - 1) = 1 + \pi' R \quad (V_0^\pi = 1)$

Basic notions, NA and NUIP

C : set of admissible strategies

- The standard admissibility condition $V_1^\pi \geq 0$ implies

$$C = \{\pi \,|\, 1 + \pi' R \geq 0\} \quad (\text{"natural constraints"})$$

- $\check{C}$: recession cone of C, namely $\check{C} = \bigcap_{a>0} a\, C$

$\rightarrow$ For natural constraints

$$\check{C} = \bigcap_{a>0} \{a\,\pi \,|\, 1 + \pi' R \geq 0\} = \bigcap_{a>0} \{\theta \,|\, a + \theta' R \geq 0\}$$

$\rightarrow$ $\check{C}$ contains strategies that can be arbitrarily scaled.

Basic notions, NA and NUIP

- Set of arbitrage opportunities

$$I := \{\pi \,|\, \pi' R \geq 0 \,\text{a.s. and } P\{\pi' R > 0\} > 0\}$$

→ For an admissibility set C, NA holds if $I \cap C = \emptyset$

Definition: NUIP (no unbounded increasing profit) holds if

$$I \cap \check{C} = \emptyset$$

- Under NUIP one may have classical arbitrage, but not with arbitrarily large profits *(no scalable arbitrage strategies)*.
- *Under natural constraints NUIP holds iff $\check{C}$ contains only the trivial strategy $\pi = 0$. Equivalently, NUIP holds whenever the strategies, that can be arbitrarily scaled, reduce to the trivial strategy.*

→ It becomes thus intuitive that, with further restrictions on the strategies, one may have classical arbitrage, but NUIP nevertheless holds.

Basic notions, tools for NUIP

$\rho \in C$ (*C may now result also from possible restrictions in addition to the natural constraints*) is called weak numeraire portfolio (WNP) if

$$E\left\{\frac{V_1^\pi}{V_1^\rho}\right\} \leq 1 = \frac{V_0^\pi}{V_0^\rho}\,, \quad \forall \pi \in C$$

$\rightarrow$ With V^ρ acting as a "numeraire", $\frac{V^\pi}{V^\rho}$ is a then supermartingale, i.e. V^ρ is a supermartingale deflator for V^π with $\pi \in C$.

If a WNP ρ satisfies $E\left\{\frac{V_1^\pi}{V_1^\rho}\right\} = 1\,, \quad \forall \pi \in C$, it is called strong numeraire portfolio (SNP)

$\rightarrow$ Whenever in this case the vectors of the canonical base in $\mathbb{R}^d$ belong to C, then $E\left\{\frac{S_1^i}{V_1^\rho}\right\} = S_0^i\,,\ i = 1, \cdots, d$, i.e. for V_1^ρ as numeraire the physical measure is then a martingale measure.

Basic results, tools for NUIP

- $\rho \in C$ is called growth optimal (GOP) if

$$E\left\{\log\left(\frac{V_1^\pi}{V_1^\rho}\right)\right\} \leq 0\,, \quad \forall \pi \in C$$

$\rightarrow$ The GOP can be obtained as the log-optimal portfolio.

It can be shown that

$$\mathit{NUIP} \quad \Leftrightarrow \quad \exists \rho \text{ WNP} \quad \Leftrightarrow \quad \rho \text{ is GOP}$$

$$(\rho \text{ a GOP+an additional condition } \Rightarrow \rho \text{ is SNP})$$

- One also has: ρ SNP $\quad \Rightarrow \quad \exists!\, Q$ EMM with $\frac{dQ}{dP} = \frac{1}{V_1^\rho}$

$\rightarrow$ If ρ is only WNP with $E\{1/V_1^\rho\} < 1$, then an EMM Q may still exist (i.e. NA may still hold) but then $\frac{dQ}{dP} \neq \frac{1}{V_1^\rho}$ *(pricing under the physical measure and with V^ρ as numeraire is then not possible).*

Example 1

One period, one risky asset

$$\begin{cases} S^0 \equiv & 1 \\ S_1 = & S_0(1+R) \quad \text{with return} \quad R = e^Y - 1\,,\ Y \sim \mathcal{N}(0,1) \\ & \Rightarrow \quad R \in (-1,+\infty);\ \text{to simplify notation put } S_0 = 1 \end{cases}$$

- $V_1^\pi = (1-\pi) + \pi\, S_1 = 1 + \pi\,(S_1 - 1) = 1 + \pi\,(e^Y - 1)$
- Natural constraints $(P\{V_1^\pi \geq 0\} = 1\ )$ are $\pi \in [0,1]$ *(short selling and borrowing is prohibited)*
- as additional constraint take $\pi \leq \bar{\pi} \in (0,1)$ so that $C = \{\pi \in [0,\bar{\pi}]$ for a given $\bar{\pi} \in (0,1)\}$

Explanation: *One cannot invest in the risky asset more than a proportion $\bar{\pi}$ of one's wealth.*

Example 1

→ NA holds irrespective of $\bar{\pi}$ $\Rightarrow$ also NUIP holds.

→ Can show that $E\{V_1^\pi\}$ is strictly increasing for $\pi \in [0, 1/2]$.

Case A For $\bar{\pi} \geq \frac{1}{2}$ *(in particular for only natural constraints)*:
V_1^ρ with $\rho = (\frac{1}{2}, \frac{1}{2})$ is SNP and $\exists$ EMM Q s.t.

$$\frac{dQ}{dP} = \frac{1}{V_1^\rho} \qquad \left(E\left\{\frac{1}{V_1^\rho}\right\} = 2\,E\left\{\frac{1}{e^Y+1}\right\} = 1 \right)$$

→ In fact, $E^Q\{S_1\} = E^P\left\{\frac{S_1}{V_1^\rho}\right\} = 2\,S_0 E^P\left\{\frac{e^Y}{1+e^Y}\right\} = S_0$.

Example 1

Case B for $\bar{\pi} < \frac{1}{2}$ implying $\rho = \bar{\pi}$. (*Recall* $E\{V_1^{\pi}\}$ *is strictly increasing for* $\pi \in [0, 1/2]$).

V_1^{ρ} is only WNP, NUIP holds, $\exists$ EMM Q, but $\frac{dQ}{dP} \neq \frac{1}{V_1^{\rho}}$ being

$$E\left\{\frac{1}{V_1^{\rho}}\right\} < 1.$$

→ A martingale measure can be shown to be given by $\frac{dQ}{dP} := e^{-\frac{Y}{2}-\frac{1}{8}}$ which implies $E^Q\left\{e^Y\right\} = E\left\{e^{\frac{Y}{2}-\frac{1}{8}}\right\} = 1$ and thus $E^Q\{S_1\} = E^Q\{S_0(1+e^Y-1)\} = S_0$

Example 1

If we had a return R with $R \geq 0$ a.s. and $P\{R > 0\} > 0$, then NA cannot hold but, independently of $\bar{\pi} \in (0, 1)$,

$$\check{C} = \cap_{a>0}\{\theta \,|\, \theta \geq 0, \text{ and } \theta \leq a\} \quad \text{reduces to} \quad \theta = 0$$

implying $I \cap \check{C} = \emptyset$ and thus that NUIP holds.

→ *ρ is only WNP*

→ NUIP holds but no EMM exists not even an ESMM.

Example 2

One period, two risky assets

$$\begin{cases} S^0 \equiv & 1 \quad \text{and, for } i = 1,2 \\ S_1^i = & S_0^i(1+R^i) \quad \text{with returns} \quad R^1 = Y + \gamma\,(Z-1),\ R^2 = Z-1 \\ & (\text{to simplify notation: } S_0^1 = S_0^2 = 1) \end{cases}$$

- Y, Z are independent on $[\underline{y}, \bar{y})$ and $[\underline{z}, \bar{z})$ respectively.
- $\gamma \in \mathbb{R}$ accounts for possible dependency between R^1 and R^2.

$\rightarrow$ In order that $S_1^i \geq 0$, $i = 1,2$ we need $R^i \geq -1$, $i = 1,2$ and so we may choose:

$$\begin{cases} \text{for } \gamma \in [0,1): & \underline{y}, \underline{z} \geq 0 \\ \text{for } \gamma \geq 1: & \underline{y} \geq 0\,,\ \underline{z} \geq \frac{\gamma-1}{\gamma} \in [0,1) \\ \text{for } \gamma < 0: & \underline{y} \geq -\gamma\bar{z} + (\gamma-1) \quad \Leftrightarrow \quad \bar{z} \leq -\frac{\underline{y}}{\gamma} + 1 - \frac{1}{\gamma} \\ & \text{i.e. } \bar{z} \leq 1 - \frac{1}{\gamma} \text{ if one wants } \underline{y} = 0 \end{cases}$$

Example 2

- Self-financing portfolio

$$V_1^\pi = 1 + \pi^1 Y + (\pi^2 + \gamma\pi^1)(Z-1)\,;\ (S_0^1 = S_0^2 = 1 \ \Rightarrow\ V_0^\pi = 1)$$

- The natural constraints $(P\{V_1^\pi \geq 0\} = 1)$ are satisfied if

$$\pi^1 \geq 0 \quad \text{and} \quad -\gamma\pi^1 \leq \pi^2 \leq 1 - \gamma\,\pi^1 \qquad \text{for } \gamma \in [0,1)$$

$$\pi^1 \geq 0 \quad \text{and} \quad -\gamma\pi^1 \leq \pi^2 \leq \gamma - \gamma\,\pi^1 \qquad \text{for } \gamma \geq 1$$

$$\pi^1 \geq 0 \quad \text{and} \quad 2\gamma - \gamma\pi^1 \leq \pi^2 \leq 2 - \gamma\,\pi^1 \quad \text{for } \gamma < 0$$

$\rightarrow$ The above conditions are also necessary if $\underline{y} = 0\,,\ \bar{y} = \infty$.

Example 2

- As additional constraint take

$$\pi^1 + \pi^2 \leq c \ , \quad c > 0$$

Explanation: *For $c > 1$, there is thus an upper limit c-1 to what can be borrowed from the bank account, while for $c \in (0, 1)$ the additional constraint imposes that at least a proportion $1 - c$ of wealth must be invested in the riskless asset.*

Example 2

NUIP property

Given the admissible set C resulting from our natural and additional constraints, one obtains

$$\check{C} = \bigcap_{a>0} a\,C = \{(\pi^1, \pi^2) \,|\, \pi^1 \geq 0, \pi^1 + \pi^2 \leq 0, -\gamma\pi^1 \leq \pi^2 \leq -\gamma\pi^1\}$$

- **Case** $\gamma < 1$: $\check{C}$ contains only the trivial strategy $\pi = 0$

$\rightarrow$ $I \cap \check{C} = \emptyset$ $\Rightarrow$ NUIP holds

- **Case** $\gamma \geq 1$: the line $\pi^2 = -\gamma\pi^1$ belongs to $\check{C}$.

$\rightarrow$ $I \cap \check{C} \neq \emptyset$ $\Rightarrow$ NUIP does not hold and thus neither NA (*shall not consider further this case.*)

Example 2

NA property

- Recalling

$$V_1^\pi = 1 + \pi^1 Y + (\pi^2 + \gamma\pi^1)(Z - 1),$$

on the line $\pi^2 = -\gamma\,\pi^1$, being $\pi^1 \geq 0$, $Y \geq 0$ a.s., we have $V_1^\pi = 1 + \pi^1 Y \geq 1 = V_0$.

- There are thus admissible arbitrage strategies on the line segment $\pi^2 = -\gamma\,\pi^1$ for the range of possible values for π^1 s.t. the constraints are satisfied. Considering, as we do now, $\gamma < 1$, this range is given by

$$0 < \pi^1 \leq \frac{c}{1-\gamma}$$

$\rightarrow$ Shall call this line segment arbitrage line.

Example 2

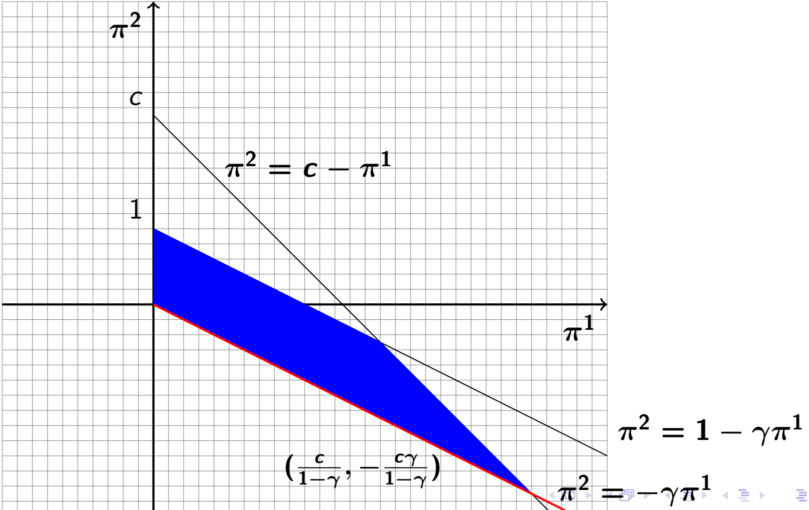
- Maximal arbitrage is obtained in correspondence to the largest admissible value of π^1, namely for the strategies

$$\begin{cases} \left(\frac{c}{1-\gamma}, -\frac{\gamma\, c}{1-\gamma}\right) & \text{for} \quad \gamma \in [0,1) \\ \left(\frac{c-2\,\gamma}{1-\gamma}, -\frac{\gamma(c-2\,\gamma)}{1-\gamma}\right) & \text{for} \quad \gamma < 0 \end{cases}$$

$\rightarrow$ NUIP holds but NA does not hold and $\nexists$ an ESMM.

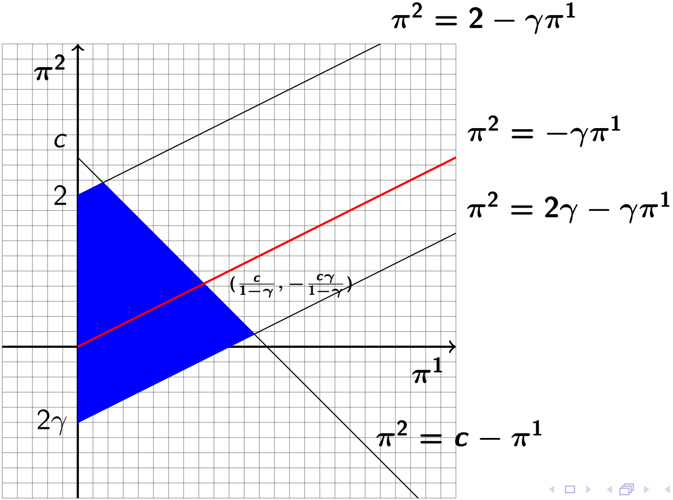
Example 2, Fig.1

$C := \{(\pi^1, \pi^2) \mid \text{natural and additional constraints are all satisfied}\}$
for the case $\gamma \in [0, 1)$ and $c > 1$



Example 2, Fig.2

$C := \{(\pi^1, \pi^2) \mid \text{natural and additional constraints are all satisfied}\}$ for the case $\gamma < 0$ and $c > 1$



Example 2

Coming to the log-optimal and thus numeraire portfolio, let

$$E\{\log(V_1^\pi)\} = E\{\log[1 + \pi^1 Y + (\gamma\,\pi^1 + \pi^2)\,(Z - 1)]\}$$

$\rightarrow$ Any optimal portfolio *(for a strictly increasing utility function)* turns out to satisfy the additional constraints as an equality, i.e. $\pi^2 = c - \pi^1$.

We may thus consider the maximization of

$$E\{\log[1 + \pi^1 Y + (\gamma\,\pi^1 + \pi^2)\,(Z - 1)]\}$$

$$= E\{\log[1 + \pi^1 Y + (\pi^1(\gamma - 1) + c)\,(Z - 1)]\}$$

in the single variable π^1 in its admissible ranges that depend on the value of γ, namely

Example 2

$$\pi^1 \in \begin{cases} \left[\left(\frac{c-1}{1-\gamma} \right)^+ , \frac{c}{1-\gamma} \right] & \text{if} \quad \gamma \in [0,1) \\ \left[\left(\frac{c-2}{1-\gamma} \right)^+ , \frac{c-2\gamma}{1-\gamma} \right] & \text{if} \quad \gamma < 0 \end{cases}$$

as it results from imposing that $(\pi^1, c - \pi^1)$ satisfies the respective natural constraints.

Example 2

The maximizing strategy and the corresponding log-optimal portfolio value depend in general on the distribution of Y and Z.

→ **Question:** will the log-optimal portfolio strategy coincide with an (maximal) arbitrage strategy?

→ Again, this depends on the distribution of Y and Z

Example 2

Case where the log-optimal portfolio (and thus GOP) strategy coincides with the maximal arbitrage strategy

Let $E\{Z\} = 1$

- Case $\gamma \in [0,1)$: By double conditioning on Y, using Jensen's inequality and $\pi^{1,max} = \frac{c}{1-\gamma}$, for any admissible $\pi^1 \in \left[\left(\frac{c-1}{1-\gamma} \right)^+, \frac{c}{1-\gamma} \right]$ one has

$$
\begin{aligned}
E\{\log(V_1^\pi)\} &= E\left[\log\left(1+\pi^1(Y+(1-\gamma)(1-Z))+c(Z-1)\right)\right\} \\
&\le E\left\{\log\left(1+\pi^1(Y+(1-\gamma)(1-E[Z|Y]))+c(E[Z|Y]-1)\right)\right\} \\
&= E\left\{\log\left(1+\pi^1 Y\right)\right] \le E\left[\log\left(1+\pi^{1,\max} Y\right)\right\} \\
&= E\left\{\log\left(1+\pi^{1,\max}(Y+(1-\gamma)(1-Z))+c(Z-1)\right)\right\}
\end{aligned}
$$

$\rightarrow$ $\pi^{\max}$ *is thus log-optimal and also GOP.*

Example 2

- Case $\gamma \in [0,1)$: By double conditioning on Y, using Jensen's inequality and $\pi^{1,max} = \frac{c}{1-\gamma}$, for any admissible $\pi^1 \in \left[\left(\frac{c-1}{1-\gamma}\right)^+, \frac{c}{1-\gamma}\right]$ one has

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\begin{aligned}
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&\leq E\left\{\log\left(1+\pi^1(Y+(1-\gamma)(1-E[Z|Y]))+c(E[Z|Y]-1)\right)\right\} \\
&= E\left\{\log\left(1+\pi^1 Y\right)\right] \leq E\left[\log\left(1+\pi^{1,\max} Y\right)\right\} \\
&= E\left\{\log\left(1+\pi^{1,\max}(Y+(1-\gamma)(1-Z))+c(Z-1)\right)\right\}
\end{aligned}
$$

$\rightarrow$ $\pi^{\max}$ *is thus log-optimal and also GOP*.

- Case $\gamma < 0$: $\pi^{\max}$ can be shown not to be log-optimal. (*Recall that in this case the admissible values for* π^1 *are* $\pi^1 \in \left[\left(\frac{c-2}{1-\gamma}\right)^+, \frac{c-2\gamma}{1-\gamma}\right]$, *a range wider than for* $\gamma \in [0,1)$).

Example 2

The log-optimal portfolio (and thus GOP) strategy does not coincide with the maximal arbitrage strategy

Let $c = 1$, $\gamma = 1/2$ with $Y \sim \mathrm{Exp}(1)$ and $Z \sim \mathrm{Exp}(\alpha)$

Recall that on the arbitrage line $\pi^2 = -\gamma\pi^1$ we have $V_1^\pi = 1 + \pi^1 Y$ so that $\pi^{1,max} = \frac{c}{1-\gamma}$. Considering then the portfolio $\pi = (0,1)$ (i.e., wealth fully invested in the second asset), it holds that

$$E\left\{\frac{V_1^\pi}{V_1^{\pi^{\max}}}\right\} = E\left\{\frac{Z}{1+2Y}\right\} = \frac{\sqrt{e}\int_{1/2}^{+\infty}\frac{e^{-t}}{t}dt}{2\alpha} > 1 \quad \text{for} \quad \alpha < 0.461$$

$\rightarrow$ $\pi^{\max}$ can thus not be the GOP (if α is sufficiently small).

- Since for any arbitrage portfolio π^a we have $V_1^{\pi^a} \leq V_1^{\pi^{\max}}$, it follows that, even in presence of arbitrage opportunities, the log-optimal portfolio is not necessarily an arbitrage portfolio.

A note on pricing

- If only NUIP holds, but not also NA, martingale pricing is not possible.
- NUIP $\rightarrow$ $\exists$GOP $(\rightarrow)$ $\exists$ log $-$optimal portfolio $\rightarrow$ log-indifference pricing is possible, namely

$$p^{log}(H) = E\left\{\frac{H}{V_1^{\rho}}\right\}$$

is by definition a "fair price", i.e. the benchmarked (in units of the GOP) price process is a martingale.

- Also superhedging pricing is possible and, if $\exists$ SNP, real world pricing. In general they are different, but coincide in complete markets.

Un bon et joyeux anniversaire, Nicole!