Bridges...
ou comment distinguer les vrais des faux

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Outline

1. Brownian Bridges and α-pinned Brownian diffusion

2. Are pinned diffusions always bridges?

Brownian Bridge as Gaussian process

Characterization through the 1st and 2nd moment

The x-y-Brownian bridge $B^{xy} = (B^{xy}_t)_{t\in[0,1]}$ is the Gaussian process with the following moments

$$\mathbf{E}\left(B^{xy}_t \right) = (1-t)\,x + t\,y$$
$$\mathbf{Cov}\left(B^{xy}_s, B^{xy}_t\right) = \min(s,t) - s\,t$$

As Gaussian Markov process

B^{xy} is the Markov process satisfying $\mathbf{P}\left(B^{xy}_0 = x \right) = 1$ whose transition probability density is

$$p^y(s,w;t,z) = \frac{p(s,w;t,z)p(t,z;1,y)}{p(s,w;1,y)},\ s<t,\ \ w,z\in\mathbb{R}$$

where $\quad p(s,w;t,z) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{1}{2}\frac{(z-w)^2}{t-s}} \quad$ is the

transition probability density of the Brownian motion.

Brownian Bridge as solution of an SDE with affine drift

Consider the stochastic differential equation

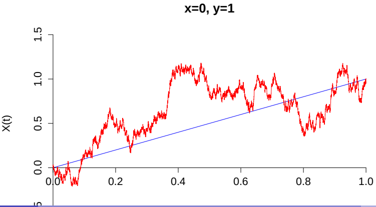
$$dX_t = \frac{y - X_t}{1-t}\, dt + dB_t, \quad X_0 = x, \tag{1}$$

where B is a real Brownian motion.

Its unique strong solution on $[0,1)$ is the x-y-Brownian Bridge given by

$$B_t^{xy} = (1-t)\,x + t\,y + (1-t)\int_0^t \frac{1}{1-s}\, dB_s, \quad t \in [0,1).$$

Extending it by continuity, one gets $B_1^{xy} = y$.



Brownian Bridge as (unique) solution of a **duality formula**

For all Φ smooth cylindrical functionals and for all **loop** $g \in L^2([0,1];\mathbb{R})$,

$$\mathbf{E}\,(\,D_g\Phi(B^{xy}\,) = \mathbf{E}\,(\,\Phi(B^{xy})\;\delta(g)\,) \tag{2}$$

- A **loop** on $[0,1]$ is a function g with vanishing integral: $\int_0^1 g(t)dt = 0$.
- $D_g\Phi$ is the Gâteaux/**Malliavin -derivative** of Φ in the direction $\int_0^{\cdot} g(t)dt$, element of the Cameron-Martin space.
- $\delta(g) := \int_0^1 g(t)\,dB_t^{xy}$ is the **stochastic integral** under the Brownian Bridge.

Remark: Set $b(t,z) := \frac{y-z}{1-t}$, the drift of the x-y-Brownian Bridge. Then

$$F_b := \partial_t b + \nabla_z b.b + \frac{1}{2}\triangle_z\, b \equiv 0.$$

as well as for any other Brownian Bridge.

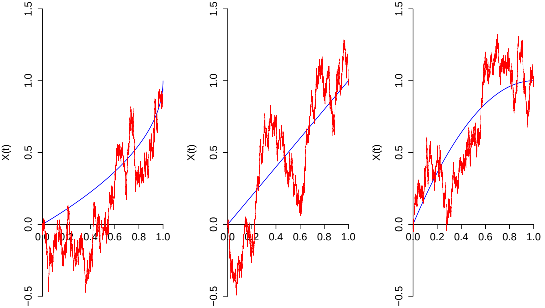
The α-pinned Brownian diffusion

- Perturbing the SDE (1), consider for $\alpha > 0$

$$dX_t = \alpha \, \frac{y - X_t}{1 - t} \, dt + dB_t, \quad X_0 = x. \tag{3}$$

Its strong sol. $B_\alpha^{xy}(\cdot)$: α-pinned Brownian diffusion between x and y.

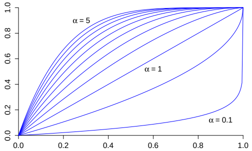
- Also called α-Wiener bridge, scaled Wiener bridge or α-Brownian bridge M. J. Brennan and E. Schwartz, *Arbitrage in stock index futures* ('90), R. Mansuy ('04), M. Barczy, G. Pap, P. Kern ('10-11), X.-M. Li ('16)



$$x = 0, \quad y = 1, \quad \alpha = 0.5, 1, 2$$

- B_α^{xy} is the Gaussian process with 1st and 2nd moments

$$\mathbf{E}\,(\,B_\alpha^{xy}(t)\,) = (1-t)^\alpha\, x + (1-(1-t)^\alpha)\, y$$



For $s \leq t$,

$$\mathbf{Cov}\,(B_\alpha^{xy}(s), B_\alpha^{xy}(t)) = \begin{cases} -\sqrt{(1-s)(1-t)}\log(1-s), & \alpha = \frac{1}{2}, \\ \dfrac{(1-s)^\alpha(1-t)^\alpha(1-(1-s)^{1-2\alpha})}{1-2\alpha}, & \alpha \neq \frac{1}{2}. \end{cases}$$

- For any $t \in [0,1)$,
$B_\alpha^{xy}(t) = (1-t)^{\alpha}\, x + (1-(1-t)^{\alpha})\ y + (1-t)^{\alpha} \int_0^t \frac{1}{(1-s)^{\alpha}}\, dB_s.$

Questions:

(i) Does this process degenerate at time 1: $\lim_{t\to 1} B_\alpha^{xy}(t) = y\ a.s.$?
(ii) Does the family $(B_\alpha^{xy}; x, y \in \mathbb{R})$ match with a family of bridges ?

More general class of (pinned) diffusions

Joint work with F. Hildebrandt (Hamburg).

Take h and σ continuous functions on $[0,1)$, $\sigma > 0$. Consider

$$dX_t = h(t)\,(y - X_t)\,dt + \sigma(t)\,dB_t, \quad X_0 = x,$$

whose solution X^{xy} satisfies for any $t \in [0,1)$,

$$X_t^{xy} = \varphi(t)\,x + (1 - \varphi(t))\;y\, + \varphi(t)\int_0^t \frac{\sigma(s)}{\varphi(s)}\,dB_s, \tag{4}$$

where $\varphi(t) := \exp(-\int_0^t h(s)ds)$.

Proposition:

Suppose (A1) h is bounded from below and $\lim_{t\to 1}\int_0^t h(r)\,dr = +\infty$

(A2) $\lim_{t\to 1}\int_0^t \sigma^2(r)\,dr < +\infty$.

Then the above diffusion X^{xy} is **pinned** i.e. $\mathbf{P}\left(\lim_{t\to 1} X_t^{xy} = y\right) = 1$.

- One controls the product $\varphi(t)\int_0^t \frac{\sigma(s)}{\varphi(s)}\,dB_s$ and its convergence to 0 by integration by parts.

Does $(X^{xy})_{x,y}$ match with the bridges of a Gaussian process?

Theorem

Assume that the processes $\{X^{xy}, x, y \in \mathbb{R}\}$ defined by (4) constitute a family of pinned diffusions. They correspond to the bridges of a non-degenerate Gaussian Markov process Z if and only if

$$\int_0^1 \sigma^2(r)\, dr < +\infty \text{ and } \quad h(t) = \frac{\sigma^2(t)}{\int_t^1 \sigma^2(r)\, dr}, \; t \in [0,1).$$

In this case the unconditioned process Z follows the dynamics $dZ_t = \sigma(t)\, dB_t$.

- **Corollary**: The α-pinned Brownian diffusion $(\alpha \neq 1)$ does not match with the bridges of any Gaussian process:

$$\sigma \equiv 1 \Rightarrow h(t) = \frac{1}{1-t}, t < 1.$$

Sketch of the proof

- Any non-degenerate continuous Gaussian Markov process Z is a shifted space-time rescaled Brownian motion:

$$Z_t \sim \mathbf{E}\,(\,Z_t\,) + u(t)\,\hat{B}_{v(t)}$$

where u and v are positive and v is non-decreasing (see Neveu, '68).

- Therefore its covariance function and the covariance function of its bridges has a very specific structure:

$$\mathbf{Cov}\left(Z_s^{xy}, Z_t^{xy}\right) = u(s)v(s)u(t)(1 - v(t)).$$

- This leads to $\Sigma = u(1) < +\infty$ and the desired form for h.

More generally: Does $(X^{xy})_{x,y}$ match with the bridges of an Itô-diffusion?

- Consider *regular* Itô-diffusions solution of SDE

$$dZ_t = b(t, Z_t)\, dt + \rho(t, Z_t)\, dB_t$$

where b and ρ are smooth as well as the transition density of Z.

Theorem

Assume that the processes $\{X^{xy}, x, y \in \mathbb{R}\}$ defined by (4) constitute a family of pinned diffusions whose drift h and diffusion coefficients σ are of class C^1 on $[0, 1)$. They correspond to the bridges of a regular Itô diffusion Z if and only if

$$\int_0^1 \sigma^2(r)\, dr < +\infty \text{ and } \quad h(t) = \frac{\sigma^2(t)}{\int_t^1 \sigma^2(r)\, dr}.$$

In this case Z is indeed Gaussian and one can choose its drift coefficient $b \equiv 0$ and its diffusion coefficient $\rho^2(t, z) \equiv \sigma^2(t)$ (as previous).

Sketch of the proof

- Consider a regular Itô-diffusion Z, given by

  $$dZ_t = b(t, Z_t)\,dt + \rho(t, Z_t)\,dB_t$$

  and define the space-time function (*reciprocal characteristics*)

  $$F_{b,\rho}(t,z) := \partial_t \frac{b}{\rho^2}(t,z) + \frac{1}{2}\partial_z\Big(\frac{b^2}{\rho^2} + \rho^2 \partial_z \frac{b}{\rho^2}\Big)(t,z).$$

- The function $F_{b,\rho}$ is indeed invariant on the whole family of bridges associated to Z (Clark, '90):
  Z and $\tilde{Z}$ share **the same bridges** if and only if their invariant coincide, that is

  $$\rho^2 \equiv \tilde{\rho}^2 \quad \text{and} \quad F_{b,\rho} \equiv F_{\tilde{b},\tilde{\rho}}.$$

- Thus one should have
  - $\rho \equiv \sigma$
  - $F_{b,\sigma} \equiv F_{\tilde{b}_y,\sigma}$ where $\tilde{b}_y(t,z) := h(t)(y-z)$ depends on y.
- In particular,

$$F_{\tilde{b}_y,\sigma}(t,z) = \frac{h'(t)\sigma^2(t) - h(t)\partial_t\sigma^2(t) - h^2(t)\sigma^2(t)}{\sigma^4(t)}\,(y-z)$$

should not depend on y. Hence,

$$h' = h^2 \ + \ h\,\partial_t \log\sigma^2.$$

which leads to the announced form.

Remark: For the α-pinned Brownian diffusion, $\tilde{b}_y(t,z) := \alpha\frac{y-z}{1-t}$ and an explicit computation gives

$$F_{\tilde{b}_y,1}(t,z) = \alpha(1-\alpha)\frac{y-z}{(1-t)^2}$$

For $\alpha \neq 1$ does not match with the bridges of any regular Itô-diffusion.

Applications/Generalisations

- Consider the diffusion X^{xy} solution of

$$dX_t = \alpha \frac{y - X_t}{(1-t)^{1+\gamma}}\, dt + dB_t, \quad X_0 = x,$$

where $\alpha > 0$ and $\gamma \geq 0$.
For $(\alpha, \gamma) \neq (1, 0)$ this pinned Brownian diffusions cannot be obtained as the bridges of any Gaussian Markov process or of any regular Itô diffusion.

- Let $f : [0, 1) \to (0, +\infty)$ be a continuous density function and let F its primitive. We consider the diffusion X^{xy} solution of

$$dX_t = \frac{f(t)}{1 - F(t)}(y - X_t)\, dt + \sqrt{f(t)}\, dB_t, \quad t \in [0, 1), \quad X_0 = x.$$

X^{00} is known in Statistics as F-Wiener bridges , see e.g. van der Vaart. $\{X^{xy}, x, y \in \mathbb{R}\}$ coincides with the bridges of the Gaussian Markov process $Z_t = \int_0^t \sqrt{f(s)}\, dB_s \overset{(d)}{=} B_{F(t)}, \quad t \in [0, 1].$