Hedging Distributions with BSDE

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Basic Definition

Let X be an $\mathcal{F}_T$-measurable r.v. on $(\Omega, \mathcal{F}, \mathbf{P})$.

1. $F_X(\cdot)$: the probability distribution function of X;
2. $F_X^{-1}(\cdot)$: the inverse of $F_X(\cdot)$; also called the quantile function of X;
3. $\Phi(\cdot)$: the standard normal distribution;
4. $X \sim \mu$: X is an $\mathcal{F}_T$-measurable random variable following the distribution μ;
5. $\mathbb{E}[f(\mu)] := \int f(x)d\mu(x)$ for a probability distribution function μ and any function f;
6. $D_t X$: the Malliavin derivative of any random variable or vector X for $t \in [0, T]$.

Problem setting

Consider a driver g and an $\mathcal{F}_T$ measurable random variable X, such that the following BSDE

$$\begin{cases} dY_t = g(t, Y_t, Z_t)dt + Z_t' dB_t, \\ Y_T = X, \end{cases} \tag{1}$$

admits a unique adapted solution $(Y(\cdot), Z(\cdot))$. For example, $X \in L^2(\mathcal{F}_T)$ and g is a Lipschitz function.

We call Y_0 the g-expectation of X, denoted by $\mathbb{E}_g[X]$.

Important Bibliography

[EPQ97] EL KAROUI, N., PENG, S. G., AND QUENEZ, M. C. (1997) Backward stochastic differential equations in finance *Mathematical Finance*, Vol. 7, pp. 1-71

Definition

A distribution μ is called hedgeable (or replicable) via BSDE (1) (or equivalently, its driver g), if there exists $(Y(\cdot), Z(\cdot))$ following the (1) with $Y_T \sim \mu$.

Efficient Hedging Problem

If μ is a hedgeable distribution via a driver g, the problem of efficient hedging μ via the driver g is

$$\inf_{X \sim \mu} \mathbb{E}_g[X]. \tag{2}$$

- We only consider square-integrable distributions μ, that is, probability distributions satisfying $\mathbb{E}[\mu^2] < \infty$. In such case, clearly we have $X \in L^2(\mathcal{F}_T)$ whenever $X \sim \mu$.
- We call a random variable X^* an optimal solution, if $X^* \sim \mu$ and $\mathbb{E}_g[X^*] = \inf_{X \sim \mu} \mathbb{E}_g[X]$.

Linear Case

Lemma 1

If $g(t,y,z) = r_t y + \theta_t z + \delta_t$, with r, θ bounded and δ is square integrable, then

$$\mathbb{E}_g[X] = \mathbb{E}[\rho_T X] - \int_0^T \mathbb{E}[\delta_s \rho_s] ds.$$

where $\rho_t = \exp\left(-\int_0^t (r_s + \frac{1}{2}\|\theta_s\|^2) ds - \int_0^t \theta_s' dB_s\right)$.

Hardy-Littlewood Inequality

If a square integrable r.v. η is atomless, then

$$\inf_{X \sim \mu} \mathbb{E}[\eta X] = \mathbb{E}[\eta X^*] = \int_0^1 F_\eta^{-1}(1-p)\mu^{-1}(p) dp,$$

where $X^* = \mu^{-1}(1 - F_\eta(\eta))$ is the unique random variable $\sim \mu$ that is anti-comonotonic with η.

Linear Case

Theorem.

With same the driver $g(t,y,z) = r_t y + \theta'_t z + \delta_t$ as above, the efficient Hedging problem has the optimal value

$$\inf_{X\sim\mu} \mathbb{E}_g[X] = \int_0^1 F_{\rho_T}^{-1}(1-p)\mu^{-1}(p)dp - \int_0^T \mathbb{E}[\delta_s \rho_s]ds,$$

where ρ_t is defined by Lemma 1, and $X^* = \mu^{-1}(1 - F_{\rho_T}(\rho_T))$ is its unique optimal solution, provided that ρ_T is atomless.

Remark

If both r_t and θ_t are determinist processes, then ρ_T is atomless.

First experiment

Concave Case as in [EPQ97]

Suppose g is concave in (y, z). Let

$$\hat{g}(t, \alpha, \beta) = \sup_{y,z} \big(g(t, y, z) - \alpha y - \beta' z\big)$$

Then

$$g(t, y, z) = \inf_{\alpha,\beta} \big(\hat{g}(t, \alpha, \beta) + \alpha y + \beta' z\big).$$

Let $h(t, y, z; \alpha, \beta) = \hat{g}(t, \alpha, \beta) + \alpha y + \beta' z$. By [EPQ97]

$$\mathbb{E}_g[X] = \sup_{(\alpha,\beta)\in\mathcal{A}} \mathbb{E}_{h(\alpha,\beta)}[X],$$

with $(\alpha, \beta) \in \mathcal{A} := \{\mathbb{E} \int_0^T g^2(t, \alpha, \beta)dt < \infty\}$. So that

$$\inf_{X\sim\mu} \mathbb{E}_g[X] = \inf_{X\sim\mu} \sup_{(\alpha,\beta)\in\mathcal{A}} \mathbb{E}_h[X] \geq \sup_{(\alpha,\beta)\in\mathcal{A}} \inf_{X\sim\mu} \mathbb{E}_h[X].$$

Second Experiment.

We consider a market when deposit and loan rates are not same

- loan rate R_t and deposit rate r_t such that $R_t \geq r_t$
- σ_t be the an invertible volatility matrix,
- θ_t be the risk premium of the market, i.e., $\theta_t = \sigma_t^{-1}(b_t - r_t\mathbf{1})$,
- All above processes are deterministic and uniformly bounded.

For a small investor, his (self-financing) wealth process Y_t with portfolio π to hedge a random payoff X follows a BSDE:

$$\begin{cases} dY_t = (r_t Y_t + \theta_t' \sigma_t' \pi_t - (R_t - r_t)(Y_t - \mathbf{1}'\pi_t)^-)dt + \pi_t' \sigma_t dB_t, \\ Y_T = X. \end{cases}$$

The driver is $g(t,y,z) = r_t y + \theta_t' z - (R_t - r_t)(y - \mathbf{1}'(\sigma_t')^{-1} z)^-$.

Comparison Result.

Theorem [Comparison Principle]

Let $(Y^i(\cdot), Z^i(\cdot))$ be the solutions of the BSDE (1) with parameters (g_i, X_i), $i = 1, 2$, respectively. If

$$X_1 \geq X_2, \quad g_1(t, Y_t^1, Z_t^1) \leq g_2(t, Y_t^1, Z_t^1), \quad \forall\, 0 \leq t \leq T;$$

or

$$X_1 \geq X_2, \quad g_1(t, Y_t^2, Z_t^2) \leq g_2(t, Y_t^2, Z_t^2), \quad \forall\, 0 \leq t \leq T;$$

then $Y_t^1 \geq Y_t^2$ for all $0 \leq t \leq T$.

- $\mathbf{D}_{1,2}$ be the set of random variables or vectors ξ, which admits a Malliavin derivative $D_t\xi$ for $t \in [0,T]$, such that

$$\|\xi\|_{1,2} = \mathbb{E}\|\xi\|^2 + \int_0^T \|D_s\xi\|^2 \, ds < +\infty.$$

By Malliavin calculus + Comparison principle,

Theorem. [EPQ97]

$Y(\cdot)$ follows BSDE with $X \in \mathbf{D}_{1,2}$.

- If $\mathbf{1}'(\sigma_t')^{-1} D_t X \geq X$, $dP \times dt$-a.s., then $Y_t \equiv \overline{Y}_t$, where $(\overline{Y}(\cdot), \overline{Z}(\cdot))$ is the solution of the linear BSDE

$$\begin{cases} d\overline{Y}_t = (R_t\overline{Y}_t + (\theta_s - (R_s - r_s)\sigma_s^{-1}\mathbf{1})\overline{Z}_t)dt + \overline{Z}_t' dB_t, \\ \overline{Y}_T = X. \end{cases}$$

Continue...

- In fact, $\overline{Y}_t \equiv \overline{\rho}_t^{-1}\mathbb{E}[\overline{\rho}_T X|\mathcal{F}_t]$, where

$$\overline{\rho}_t = \exp\Bigg(- \int_0^t \left(R_s + \tfrac{1}{2}\|\theta_s - (R_s - r_s)\sigma_s^{-1}\mathbf{1}\|^2\right) ds - \int_0^t (\theta_s - (R_s - r_s)\sigma_s^{-1}\mathbf{1})' dB_s \Bigg). \tag{3}$$

- If $\mathbf{1}'(\sigma_t')^{-1}D_t X \le X$, $dP \times dt$-a.s., then $Y_t \equiv \widetilde{Y}_t$, where $(\widetilde{Y}(\cdot), \widetilde{Z}(\cdot))$ is the solution of the linear BSDE

$$\begin{cases} d\widetilde{Y}_t = (r_t\widetilde{Y}_t + \theta_t'\widetilde{Z}_t)dt + \widetilde{Z}_t' dB_t, \\ \widetilde{Y}_T = X. \end{cases}$$

In fact, $\widetilde{Y}_t \equiv \rho_t^{-1}\mathbb{E}[\rho_T X|\mathcal{F}_t]$.

Proposition.

With same assumptions,

$$\inf_{X\sim\mu} \mathbb{E}_g[X] \le \min\left\{\mathbb{E}[\rho_T\widetilde{X}], \mathbb{E}[\overline{\rho}_T\overline{X}]\right\},$$

where ρ_t and $\overline{\rho}_t$ are defined as above, $\widetilde{X} = \mu^{-1}(1 - F_{\rho_T}(\rho_T))$ and $\overline{X} = \mu^{-1}(1 - F_{\overline{\rho}_T}(\overline{\rho}_T))$. Moreover,

- if $\mathbf{1}'\sigma_t^{-1}D_t\widetilde{X} \le \widetilde{X}$, $dP \times dt$-a.s., then

$$CaseI: \quad \inf_{X\sim\mu} \mathbb{E}_g = \mathbb{E}_g[\widetilde{X}] = \mathbb{E}[\rho_T\widetilde{X}],$$

- if $\mathbf{1}'\sigma_t^{-1}D_t\overline{X} \ge \overline{X}$, $dP \times dt$-a.s., then

$$CaseII: \quad \inf_{X\sim\mu} \mathbb{E}_g[X] = \mathbb{E}_g[\overline{X}] = \mathbb{E}[\overline{\rho}_T\overline{X}],$$

Theorem.

For all $0 \leq t \leq T$

- If $\mu(0-) = 0$ and $\mathbf{1}'(\sigma_t')^{-1}\theta_t \leq 0$, then Case I holds.
- If $\mu(0) = 1$ and $\mathbf{1}'(\sigma_t')^{-1}(\theta_t - (R_t - r_t)\sigma_t^{-1}\mathbf{1}) \geq 0$, then Case II holds.
- Let $f(x) = \mu^{-1}(1 - F_{\rho_T}(x))$ and $c = \sup_{t\in[0,T]} 1'(\sigma_t')^{-1}\theta_t$, if

$$f(x) \geq -cxf_x(s), x > 0$$

then Case I holds.

Proof. Chain rule for Malliavin Calculus.

Remark.

If we take $\sigma_t = \begin{pmatrix} 4 & 0.1 \\ 0.1 & 1 \end{pmatrix}, \theta_t = \begin{pmatrix} 1.9 \\ 0.1 \end{pmatrix}$, then $\mathbf{1}'(\sigma_t')^{-1}\theta_t \leq 0$. In fact

$$\begin{aligned} \mathbf{1}'(\sigma_t')^{-1}\theta_t \leq 0 \quad &\Leftrightarrow \quad \mathbf{1}'(\sigma_t'\sigma_t)^{-1}(b_t - r_t) \leq 0, \\ \mathbf{1}'(\sigma_t')^{-1}(\theta_t - (R_t - r_t)\sigma_t^{-1}\mathbf{1}) \geq 0 \quad &\Leftrightarrow \quad \mathbf{1}'(\sigma_t'\sigma_t)^{-1}(b_t - R_t) \geq 0. \end{aligned}$$

Another Point of View

Special cases.

There are two important examples in theory of g-expectation:

$$g^k(z) = k|z|, g^{-k}(z) = -k|z|.$$

By comparison principle, if $g(z)$ satisfies Lipschitz condition for some constant k, then

Maybe some boundary for efficient hedging problem

$$\mathbb{E}_{g^k}[X] \leq \mathbb{E}_g[X] \leq \mathbb{E}_{g^{-k}}[X]$$

$$\Rightarrow \inf_{X\sim\mu} \mathbb{E}_{g^k}[X] \leq \inf_{X\sim\mu} \mathbb{E}_g[X] \leq \inf_{X\sim\mu} \mathbb{E}_{g^{-k}}[X]$$

Consider normal distribution $\mu = N(m, \sigma)$. For $a \geq 1$, let $\xi^a = m + \sqrt{\sigma a} B_{\frac{1}{a}} \sim N(m, \sigma)$, then

$$Z_t^{\xi^a} = \sqrt{\sigma a} 1_{\{t \leq \frac{1}{a}\}}.$$

So

$$\mathbb{E}[\xi] \leq \inf_{\xi \sim \mu} Y_0^{\xi} \leq \inf_{a \geq 1} Y_0^{\xi^a} = \inf_{a \geq 1} \left(\mathbb{E}\left[\xi\right] + k\sqrt{\frac{\sigma}{a}} \right) = \mathbb{E}\left[\xi\right].$$

Proposition

If $\mathbb{E}[\mu^2] < +\infty$, then

$$\inf_{X \sim \mu} \mathbb{E}_{g^{-k}}[X] = \mathbb{E}\left[\mu\right].$$

Sketch of Proof.

For a distribution μ, set $X = \mu^{-1}\left(\Phi_T\left(B_T\right)\right) \sim \mu$. By non-linear Feymann-Kac formula, and scale changing of Brownian motion $X^\alpha = \mu^{-1}\left(\Phi_T\left(\sqrt{\alpha}B_{\frac{T}{\alpha}}\right)\right) \sim \mu$, for $\alpha \geq 1$, we can construct a sequence $\overline{Y}_t^\alpha$, such that

$$\mathbb{E}_{g^{-k}}[X] = \overline{Y}_0^\alpha = \mathbb{E}[X] + \frac{1}{\sqrt{\alpha}}\mathbb{E}[\int_0^T k\left|\overline{Z}_s^\alpha\right| ds].$$

We can prove that $\overline{Z}_s^\alpha$ is uniformly square integrable, which yields

$$\inf_{X\sim\mu} \mathbb{E}_g[X] \leq \lim_{\alpha\to\infty} \overline{Y}_0^\alpha \leq \mathbb{E}[X] + \lim_{\alpha\to\infty} \frac{\sqrt{T}}{\sqrt{\alpha}}\left(C\mathbb{E}[X^2]\right)^{\frac{1}{2}} = \mathbb{E}[X].$$

Obviously $\mathbb{E}_{g^{-k}}[X] \geq \mathbb{E}[X]$ by definition of g-expectaion.

Corollary

For any Lipschitz function $g(z)$, efficient hedging problem is majored by

$$\inf_{X\sim\mu} \mathbb{E}_g[X] \leq \mathbb{E}\left[\mu\right].$$

Small Generalization

Assumption

The driver $g(t,z)$ is a deterministic function of (t,z), and satisfies

- Non-positivity: $g(t,z) \leq 0$;
- Lipschitz continuity: $|g(t,z_1) - g(t,z_2)| \leq K\|z_1 - z_2\|$, $K > 0$;
- Positive homogeneousity: $g(t,\alpha z) = \alpha g(t,z)$, for any $\alpha \geq 0$.

Remark

The drivers $g(t,z)$ that satisfy assumption must be of the form

$$g(t,z) \equiv -A_t' z^+ - C_t' z^-.$$

Theorem.

If a driver g satisfies above assumptions. Then

$$\inf_{X \sim \mu} \mathbb{E}_g[X] = \mathbb{E}[\mu],$$

Consider normal distribution $\mu = N(m, \sigma)$. Let $\xi^n = m + \sqrt{\sigma a} B_{\frac{T}{a}} \sim N(m, \sigma)$, for $a \geq 1$, then

$$Z_t^{\xi^a} = \sqrt{\sigma a} 1_{\{t \leq \frac{1}{a}\}}.$$

So

$$\inf_{\xi \sim \mu} Y_0^{\xi} \leq \inf_{a \geq 1} Y_0^{\xi^a} = \inf_{a \geq 1} \left(\mathbb{E}[\xi] - k\sqrt{\frac{\sigma}{a}} \right) = \mathbb{E}[\xi] - k\sqrt{\sigma}.$$

Proposition.

Given a distribution μ, set $X^* = \mu^{-1}(\Phi_T(B_T)) \sim \mu$. We consider $X^\alpha = \mu^{-1}\left(\Phi_T\left(\sqrt{\alpha} B_{\frac{T}{\alpha}}\right)\right) \sim \mu$, for $\alpha \geq 1$, then

$$\inf_{X \sim \mu} \mathbb{E}_{g^k}[X] \leq \inf_{\alpha} \mathbb{E}_{g^k}[X^\alpha] = \mathbb{E}_{g^k}[X^*] \leq \mathbb{E}[\mu].$$

Definition [Law-invariant g-expectation]

Given a distribution μ, we call a driver g is μ-invariant, if $\mathbb{E}_g[X]$ are the same, denoted as $\mathbb{E}_g[\mu]$ for all $X \sim \mu$. A g-expectation (or its driver g) is called law-invariant if it is μ-invariant for any square-integrable μ.

Example. Entropy measure

The driver $g(t, y, z) \equiv -\frac{1}{2}z^2$ is law-invariant. By Itô's formula

$$d\, e^{Y_t} = e^{Y_t} Z_t' dB_t.$$

Hence $e^{Y_0} = \mathbb{E}[e^{Y_T}] = \mathbb{E}[e^{\mu}]$ if $Y_T \sim \mu$. In another word, $\mathbb{E}_g[X] = \log\left(\mathbb{E}[e^{\mu}]\right)$ for any $X \sim \mu$, which is μ-invariant.

Law-invariant g-expectation

Theorem. [when g not depending on t]

Each time-invariant driver of the form $g(t, y, z) \equiv -f(y)\|z\|^2$ is law-invariant and

$$\mathbb{E}_g[\mu] = \varphi^{-1}\Big(\mathbb{E}[\varphi(\mu)]\Big),$$

for any distribution μ such that the right hand side is well-defined, where

$$\varphi(x) = \int_0^x \exp\left(2\int_0^y f(s)ds\right)dy.$$

- Experiment of g may depend on t is still ongoing.....
- As well as applications in portfolio selection problem....

Thanks you!

Happy Birthday!

福如东海

寿比南山