On two stochastic modelling issues

Denis Talay, INRIA Sophia Antipolis

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Outline

1. Sensitivity w.r.t. Hurst parameter of functionals of diffusions driven by fractional Brownian motions
2. On a new Wasserstein type distance
3. Conclusion

Outline

Objective: Sensitivity analysis w.r.t. the long-range/memory noise parameter for probability distributions of functionals of solutions to SDEs.

Given $H \in (\frac{1}{4}, 1)$:

$$X_t^H = x_0 + \int_0^t b(X_s^H)\ \mathrm{d}s + \int_0^t \sigma(X_s^H) \circ \mathrm{d}B_s^H,$$

where the stochastic integral is to be precised.
For $H = \frac{1}{2}$, classical MARKOVIAN SDE in the Stratonovich sense:

$$X_t = x_0 + \int_0^t b(X_s)\ \mathrm{d}s + \int_0^t \sigma(X_s) \circ \mathrm{d}B_s$$

Two sensitivity problems when the Hurst parameter H of the noise tends to the critical Brownian parameter $H = \frac{1}{2}$ from above OR from below :

- Smooth functionals: Time marginal probability distributions of X^H.
- Irregular functionals: Laplace transforms of the first passage times of X^H at given thresholds.

Motivations: neuroscience, risk analyses, stochastic computational models, etc.

Smooth functionals: Sensitivity of marginal distributions

Theorem 1 (with Alexandre Richard, ECP)

Suppose that b and σ are smooth enough and σ is strongly elliptic. Suppose that φ is bounded and Hölder continuous of order $2+\beta$ for some $\beta > 0$. Then, for any $T > 0$, there exists $C_T > 0$ such that for any $H \in [\frac{1}{4}, 1)$:

$$\sup_{t\in[0,T]} \left|\mathbb{E}\varphi(X_t^H) - \mathbb{E}\varphi(X_t)\right| \leq C_T \left|H - \tfrac{1}{2}\right|$$

Irregular functionals: Sensitivity of first exit time Laplace transforms

Theorem 2 (with Alexandre Richard, ECP)

Let $x_0 < 1$. There exists a constant $\lambda_0 \geq 1$ (depending on b and σ only) such that: for any $\epsilon \in (0, \frac{1}{4})$, there exist $\alpha > 0$ and $C_\epsilon > 0$ such that

$$\forall H \in (\tfrac{1}{4}, 1),\ \forall \lambda \geq \lambda_0, \quad \left| \mathbb{E}\left(e^{-\lambda \tau_H^X}\right) - \mathbb{E}\left(e^{-\lambda \tau_{\frac{1}{2}}^X}\right) \right| \leq C_\epsilon (1+\lambda)^{\frac{3}{2}} |H - \tfrac{1}{2}|^{\frac{1}{2} \wedge H - \epsilon}\ e^{-\alpha S(1-x_0) T(\lambda)},$$

where $S, T : \mathbb{R}_+ \to \mathbb{R}_+$ are explicit increasing functions.

Technical difficulties: accurate controls w.r.t. $H - \frac{1}{2}$, $1 - x_0$, λ.

Reminder: for $H = \frac{1}{2}$, X^H=BM and the Laplace transform is $e^{-(1-x_0)\sqrt{2\lambda}}$. Thus one needs $S(x) \simeq x$ and $T(\lambda) \simeq \sqrt{\lambda}$.

Covariance of Fractional BMs

fBM with Hurst parameter $H \in (0,1)$: self-similar Gaussian process with stationary increments and covariance function

$$R_H(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}).$$

For $\theta > \sigma > 0$ set

$$K_H(\theta,\sigma) := c_H \left\{ \left(\frac{\theta(\theta-\sigma)}{\sigma}\right)^{H-\frac{1}{2}} - (H-\tfrac{1}{2})\sigma^{\frac{1}{2}-H}\int_\sigma^\theta u^{H-\frac{3}{2}}(u-\sigma)^{H-\frac{1}{2}}\,\mathrm{d}u \right\}$$

Then, for some explicit constant c_H,

$$R_H(s,t) = \int_0^{s\wedge t} K_H(s,u)K_H(t,u)\,\mathrm{d}u.$$

Volterra representation:

For some standard Brownian motion $B \equiv B^{\frac{1}{2}}$,

$$\forall t \geq 0, \quad B_t^H = \int_0^t K_H(t,u)\,\mathrm{d}B_u$$

Malliavin calculus

Set

$$K_H^*\varphi(s) = K_H(T,s)\varphi(s) + (H-\tfrac{1}{2})c_H \int_s^T \left(\frac{\theta}{s}\right)^{H-\frac{1}{2}} (\theta-s)^{H-\frac{3}{2}} (\varphi(\theta)-\varphi(s))\,\mathrm{d}\theta$$

The Cameron-Martin space $\mathcal{H}_H$ is defined as the completion of the space of simple functions for the scalar product

$$\langle \varphi, \psi \rangle_{\mathcal{H}_H} = \langle K_H^*\varphi, K_H^*\psi \rangle_{L^2[0,T]}$$

Define the Malliavin derivative on $\mathcal{H}_H$ by

$$D^H := (K_H^*)^{-1} D$$

and the Skorokhod integral $\delta_H^{(T)}$ by duality:

$$\mathbb{E}\left(\langle u, D^H F \rangle_{\mathcal{H}_H}\right) = \mathbb{E}\left(F\delta_H^{(T)}(u)\right)$$

One has:

$$\delta_H^{(T)}(u) = \delta(K_H^* u)$$

for any u such that $K_H^* u \in \operatorname{dom} \delta$.

Stratonovich integrals

Except when $H = \frac{1}{2}$, the fBm is not a semimartingale.
The Stratonovich integral $\int u \circ \mathrm{d}B^H$ is the limit in probability (if it exists) of

$$\frac{1}{2\epsilon} \int_0^T u_s (B^H_{(s+\epsilon)\wedge T} - B^H_{(s-\epsilon)\vee 0}) \mathrm{d}s$$

when $\epsilon \to 0$.
If $u \in \mathbb{D}^{1,2}(|\mathcal{H}_H|)$ and if the following limit in probability exists:

$$\operatorname{Tr} D^H u := \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^T \langle D^H_{\cdot} u_s, \mathbf{1}_{[s-\epsilon, s+\epsilon] \cap [0,T]} \rangle_{\mathcal{H}_H} \mathrm{d}s$$

then the Stratonovich integral exists and

$$\int_0^T u_s \circ \mathrm{d}B^H_s = \delta_H(u) + \operatorname{Tr} D^H u$$

(see Nualart).

Strong solutions to SDEs

$$X_t^H = x_0 + \int_0^t b(X_s^H)\,\mathrm{d}s + \int_0^t \sigma(X_s^H) \circ \mathrm{d}B_s^H$$

- For $H > \frac{1}{2}$: solutions in the sense of Young (Nualart and Rascanu).
- For $H \in (\frac{1}{4}, \frac{1}{2})$: solutions in the sense of Alòs, León, Nualart.

Proposition (Lamperti transform)

Let $H \in (\frac{1}{4}, 1)$. Suppose b, σ smooth enough and $\sigma(x) \geq \sigma_0 > 0$.
Let $F(x) := \int_0^x \frac{1}{\sigma(z)}\,\mathrm{d}z$ and $\tilde{b} = \frac{b \circ F^{-1}}{\sigma \circ F^{-1}}$. Then $Y^H := F(X^H)$ solves

$$Y_t^H = F(x_0) + B_t^H + \int_0^t \tilde{b}(Y_s^H)\,\mathrm{d}s$$

Sobolev regularity – Itô's formula

Y^H belongs to $\mathbb{D}^{1,2}(|\mathcal{H}_H|)$ and

$$D_r^H Y_t^H = \mathbf{1}_{[0,t]}(r) \exp\left(\int_r^t b'(Y_\theta^H)\mathrm{d}\theta\right)$$

For any continuously differentiable function g, $\mathrm{Tr}\, D^H g(Y^H)$ exists and

$$\begin{aligned} G(Y_t^H) &= G(Y_0^H) + \int_0^t G'(Y_s^H)\tilde{b}(Y_s^H)\mathrm{d}s + \int_0^t G'(Y_s^H) \circ \mathrm{d}B_s^H \\ &= G(Y_0^H) + \int_0^t G'(Y_s^H)\tilde{b}(Y_s^H)\mathrm{d}s + \delta_H\left(\mathbf{1}_{[0,t]}(\cdot)G'(Y_\cdot^H)\right) \\ &\quad + \mathrm{Tr}\left[D^H G'(Y^H)\right]_t , \end{aligned}$$

where $\mathrm{Tr}\left[D^H u\right]_t := \mathrm{Tr}\, D^H\left(\mathbf{1}_{[0,t]}u\right)$.

Sensitivity of marginal distributions

Theorem 1

Suppose that b and σ are smooth enough, σ is strongly elliptic and φ is bounded and Hölder continuous of order $2+\beta$ for some $\beta>0$. Then, for any $T>0$, there exists $C_T>0$ such that for any $H\in[\frac{1}{4},1)$:

$$\sup_{t\in[0,T]}\left|\mathbb{E}\varphi(X_t^H)-\mathbb{E}\varphi(X_t)\right|\leq C_T\left|H-\tfrac{1}{2}\right|$$

Sketch of the proof of Theorem 1 ($H > \frac{1}{2}$)

Consider the parabolic PDE with initial condition φ at time $t \in (0, T]$:

$$\begin{cases} \frac{\partial}{\partial s} u(s,x) + \tilde{b}(x) \frac{\partial}{\partial x} u(s,x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} u(s,x) = 0, \quad (s,x) \in [0,t) \times \mathbb{R}, \\ u(t,x) = \varphi(x), \quad x \in \mathbb{R}. \end{cases}$$

Reminder:
$u(s, Y_s^{1/2})$ is a martingale

By our Itô's formula,

$$u(t, Y_t^H) = u(0, x_0) + \int_0^t \left(\partial_s u(s, Y_s^H) + \partial_x u(s, Y_s^H)\ \tilde{b}(Y_s^H) \right) \mathrm{d}s$$
$$+ \int_0^t \partial_x u(s, Y_s^H) \circ \mathrm{d}B_s^H.$$

Use the formula Stratonovich $\longrightarrow$ Skorokhod:

$$\begin{aligned} u(t, Y_t^H) = u(0, x_0) &+ \int_0^t \Big(\partial_s u(s, Y_s^H) + \partial_x u(s, Y_s^H)\ \tilde{b}(Y_s^H) \Big)\ \mathrm{d}s \\ &+ \delta_H \left(\mathbf{1}_{[0,t]} \partial_x u(\cdot, Y_\cdot^H) \right) \\ &+ \alpha_H \int_0^t \int_0^s |r-s|^{2H-2} D_r^H Y_s^H\ \partial_{xx}^2 u(s, Y_s^H)\ \mathrm{d}r\ \mathrm{d}s. \end{aligned}$$

Use the PDE solved by u and the fact that the Skorokhod integral has zero mean :

$$\begin{aligned} \mathbb{E}\varphi(Y_t^H) - \mathbb{E}_{x_0}\varphi(Y_t) = \mathbb{E} &\int_0^t \partial_{xx}^2 u(s, Y_s^H) \left(Hs^{2H-1} - \tfrac{1}{2} \right) \mathrm{d}s \\ &+ \alpha_H \mathbb{E} \int_0^t \int_0^s |r-s|^{2H-2} (D_r^H Y_s^H - 1) \partial_{xx}^2 u(s, Y_s^H)\ \mathrm{d}r\mathrm{d}s \\ =:&\ \Delta_H^1 + \Delta_H^2. \end{aligned}$$

For $H > \frac{1}{2}$:

We bound $|\Delta_H^1|$ as follows:

$$|\Delta_H^1| = \left| \int_0^t \partial_{xx}^2 u(s, Y_s^H) \left(Hs^{2H-1} - \tfrac{1}{2} \right) \, \mathrm{d}s \right| \leq C \left(H - \tfrac{1}{2} \right)$$

To bound $|\Delta_H^2|$, use the above formula for $D_r^H Y_t^H$:

$$\begin{aligned} |\Delta_H^2| &\leq C(H - \tfrac{1}{2}) \int_0^t \int_0^s (s-r)^{2H-2} \, (s-r) \, |\partial_{xx}^2 u(s, Y_s^H)| \, \mathrm{d}r\mathrm{d}s \\ &\leq C(H - \tfrac{1}{2}) \|\partial_{xx}^2 u\|_\infty \int_0^t \int_0^s (s-r)^{2H-1} \, \mathrm{d}r\mathrm{d}s \\ &\leq C(H - \tfrac{1}{2}). \end{aligned}$$

For $H < \frac{1}{2}$: Similar calculations.

Theorem 2

Let $x_0 < 1$. There exists a constant $\lambda_0 \geq 1$ (depending on b and σ only) such that: for any $\epsilon \in (0, \frac{1}{4})$, there exist $\alpha > 0$ and $C_\epsilon > 0$ such that

$$\forall H \in (\tfrac{1}{4}, 1),\ \forall \lambda \geq \lambda_0, \quad \left| \mathbb{E}\left(e^{-\lambda \tau_H^X}\right) - \mathbb{E}\left(e^{-\lambda \tau_{\frac{1}{2}}^X}\right) \right| \leq C_\epsilon (1+\lambda)^{\frac{3}{2}} |H - \tfrac{1}{2}|^{\frac{1}{2} \wedge H - \epsilon}\ e^{-\alpha S(1-x_0) T(\lambda)},$$

where $S, T : \mathbb{R}_+ \to \mathbb{R}_+$ are explicit increasing functions.

Sketch of the proof of Theorem 2

The proof notably uses estimates and tricks from Nualart and Rascanu (2002), Alòs and Nualart(2003), Peccati, Thieullen, and Tudor (2006), Nualart (2006).
After having used Lamperti transform,

$$\mathbb{E}\left(e^{-\lambda\tau_H^X(x_0)}\right) - \mathbb{E}\left(e^{-\lambda\tau_{\frac{1}{2}}^X(x_0)}\right) = \mathbb{E}\left(e^{-\lambda\tau_H^Y(F(x_0))}\right) - \mathbb{E}\left(e^{-\lambda\tau_{\frac{1}{2}}^Y(F(x_0))}\right).$$

Set $y_0 := F(x_0)$ and $\Theta := F(1)$.

$w_\lambda(y_0) := \mathbb{E}\left(e^{-\lambda \tau^Y_{1/2}(y_0)}\right)$ satisfies

$$\begin{cases} \tilde{b}(y) w'_\lambda(y) + \frac{1}{2} w''_\lambda(y) & = \lambda w_\lambda(y), \;\; y < \Theta \;, \\ w_\lambda(\Theta) & = 1 \;, \\ \lim\limits_{y \to -\infty} w_\lambda(y) & = 0 \;. \end{cases}$$

Apply Itô's formula: for any $0 < t \leq \tau^Y_H \wedge T$,

$$\begin{aligned} e^{-\lambda t} w_\lambda(Y^H_t) - w_\lambda(y_0) = \int_0^t e^{-\lambda s} \left(w'_\lambda(Y^H_s) \tilde{b}(Y^H_s) - \lambda w_\lambda(Y^H_s) \right) \, \mathrm{d}s \\ + \delta_H(\mathbf{1}_{(0,t)} w'_\lambda(Y^H)) \\ + \mathrm{Tr}\left[D^H w'_\lambda(Y^H) \right]_t . \end{aligned}$$

For any continuously differentiable function g we have

$$\mathrm{Tr}\left[D^H g(Y^H)\right]_t = \int_0^t \left(Hs^{2H-1} + \int_0^s K_H^* Z(s,\cdot)(u) \partial_s K_H(s,u) \mathrm{d}u \right) \mathrm{d}s$$

with

$$Z(s,u) := D_u^H Y_s^H - \mathbf{1}_{[0,s]}(u)$$

Set

$$\Delta(s,H) := Hs^{2H-1} - \frac{1}{2} + \int_0^T K_H^* Z(s,\cdot)(u) \partial_s K_H(s,u) \mathrm{d}u$$

We get

$$\mathbb{E}\left(w_\lambda(Y^H_{T\wedge\tau_H})\ e^{-\lambda(T\wedge\tau_H)} \right) - w_\lambda(y_0) = \mathbb{E}\left[\int_0^{T\wedge\tau_H} \Delta(s,H) w''_\lambda(Y_s^H)\ e^{-\lambda s}\ \mathrm{d}s \right] + \mathbb{E}\left[\delta_H^{(T)} \left(\mathbf{1}_{[0,t]}\ w'_\lambda(Y^H_\cdot)\ e^{-\lambda\cdot} \right)\Big|_{t=T\wedge\tau_H} \right]$$

Finally,

$$\begin{aligned}\mathbb{E}\left(e^{-\lambda\tau_H}\right) - \mathbb{E}\left(e^{-\lambda\tau_{\frac{1}{2}}}\right) &= \mathbb{E}\left[\int_0^{\tau_H} \Delta(s,H)e^{-\lambda s} w_\lambda''(Y_s^H)\,\mathrm{d}s\right] \\ &\quad + \lim_{T\to+\infty} \mathbb{E}\left[\delta_H^{(T)}\left(\mathbf{1}_{[0,t]} e^{-\lambda\cdot} w_\lambda'(Y_\cdot^H)\right)\Big|_{t=\tau_H\wedge T}\right] \\ &=: I_1(\lambda) + I_2(\lambda).\end{aligned}$$

To estimate $I_1(\lambda)$: A crucial lemma

For any $H \in (\frac{1}{4}, 1)$,

$$\left| \int_0^s K_H^* Z(s,\cdot)(u) \partial_s K_H(s,u) \, \mathrm{d}u \right| \leq |H - \tfrac{1}{2}| e^{s\|\tilde{b}'\|_\infty} (1 + Cs^2) \quad \text{a.s.}$$

To estimate $I_2(\lambda)$

Define the field $\{U_t(v), t \in [0,N], v \geq 0\}$ and the process $\{\Upsilon_t, t \in [0,N]\}$ by

$$\forall t \in [0,N],\ \ U_t(v) = \{K_H^* - \mathrm{Id}\}\left(\mathbf{1}_{[0,t]}(\bullet)\ w_\lambda'(BY_\bullet{}^H)\ e^{-\lambda \bullet}\right)(v),$$

and (notice the subscript for Skorokhod integrals on $[0,N]$)

$$\Upsilon_t = \delta^{(N)}(U_t(\bullet)).$$

Key lemma:
For any $\epsilon \in (0,\frac{1}{2})$, there exist constants $C, \alpha > 0$ such that

$\forall n \in \mathbb{N},\ \ \forall H \in (\frac{1}{4}, 1),$

$$\mathbb{E} \sup_{t \in [n,n+1]} \left[\mathbf{1}_{\{\tau_H \geq t\}} |\Upsilon_t - \Upsilon_n|\right] \leq C(H)\ |H - \tfrac{1}{2}|^{\frac{1}{2} \wedge H(1-2\epsilon)}\ e^{-\frac{\lambda n}{12}} e^{-\frac{\alpha}{2}\eta S(\Theta - y_0) T(\lambda)}.$$

Two open questions

Notice that the (pure Brownian) Markov model is robust.

Open questions:

- Get rid of the ellipticity condition: Change the PDE. But ...
- Examine multi-dimensional models. But ...
- Invert Laplace transforms and get information on the robustness of densities

Outline

One possible strategy for singular models: regularization

Motivation 1: Modelling issue. Given an exact diffusion model, how to select a simplified diffusion model within a class of admissible models under the constraint that the probability distribution of the exact model is preserved as much as possible?
Motivation 2: Remove singularities. Regularization of interaction kernels, simulation of truncated laws, etc. Not too much! since explosion times are crucial informations.

Proposed solution: Introduce a Wasserstein-type distance on the set of the probability distributions of strong solutions to SDEs which can easily be estimated and computed.

Strategy: Minimize the distance between probability distributions.Restrict the set of possible coupling measures.

Standard Wasserstein distance

On the set of probability measures on $(L_2([0,T],\mathbb{R}^d), d_2)$ let $\mathcal{W}^2$ be the standard Wasserstein distance

$$\boxed{\mathcal{W}^2(\mathbb{P};\overline{\mathbb{P}}) := \left\{ \inf_{\pi\in\Pi(\mathbb{P};\overline{\mathbb{P}})} \int_\Omega \int_0^T |\omega_s - \overline{\omega}_s|^2 \, ds \; \pi(d\omega, d\overline{\omega}) \right\}^{\frac{1}{2}},}$$

where $\Pi(\mathbb{P};\overline{\mathbb{P}})$ is the set of all the probability distributions π on $L_2([0,T],\mathbb{R}^{2d})$ with marginal distributions $\mathbb{P}$ and $\overline{\mathbb{P}}$.

This distance metrizes the weak topology on the set of probability measures π on $(L_2([0,T],\mathbb{R}^d), d_2)$ such that $\mathbb{E}^\pi \int_0^T |\omega_s|^2 \, ds < \infty$.

Unfortunately, the numerical computation of $\mathcal{W}^2$ or any other Wasserstein distance on an infinite dimensional space is impossible .

Let $\Omega := \mathcal{C}(0, T; \mathbb{R}^d)$ equipped with the canonical filtration $(\mathcal{F}_s, 0 \leq s \leq T)$ and Borel σ-field $\mathcal{F} := \bigvee_{0 \leq s \leq T} \mathcal{F}_s$.

Definition

Let $\mathbf{P}$ be the set of probability measures $\mathbb{P}$ on Ω s.t.: there exist x_0 in $\mathbb{R}^d$ and bounded Lipschitz applications μ and σ satisfying

$$\exists \lambda > 0,\ \forall 0 \leq s \leq T, \forall \xi \in \mathbb{R}^d, \forall x \in \mathbb{R}^d,\ \sum_{i,j=1}^{d} (\sigma(x)\sigma(x)^{\intercal})^{ij} \xi^i \xi^j \geq \lambda |\xi|^2,$$

such that $\mathbb{P} \equiv \mathbb{P}_{x_0}^{\mu,\sigma}$ is the probability distribution of the unique strong solution to the SDE with coefficients μ and σ and initial condition x_0.

Main issue: Any coupling measure of two probability distributions in $\mathbf{P}$ cannot be represented as the probability distribution of a solution of a $2d$-dimensional SDE. We thus modify the definition of $\mathcal{W}^2$ distance by restricting the set of coupling measures .

Our admissible couplings

Definition

Given $\mathbb{P}^{\mu,\sigma}_{x_0}$ and $\mathbb{P}^{\overline{\mu},\overline{\sigma}}_{\overline{x}_0}$ in $\mathbf{P}$, let $\widetilde{\Pi}(\mathbb{P}^{\mu,\sigma};\mathbb{P}^{\overline{\mu},\overline{\sigma}})$ be the set of the probability laws $\widetilde{\mathbb{P}}$ on $(\Omega^{\otimes 2},\mathcal{F}^{\otimes 2})$ s.t.

(i) On some probability space equipped with independent Brownian motions $(W,\overline{W})$ and some filtration $\mathcal{G}$, there exist $\mathcal{G}$-predictable processes $(\mathcal{C}_t)$ and $(\mathcal{D}_t)$ taking values in the space of correlation matrices $\mathbf{C}_d$ and an $\mathcal{G}$-adapted solution $(X^{\mathcal{C}},\overline{X})$ to

$$\boxed{\begin{cases} X^{\mathcal{C}}_t = x_0 + \int_0^t \mu(X^{\mathcal{C}}_s)\,ds + \int_0^t \sigma(X^{\mathcal{C}}_s)\,(\mathcal{C}_s\,d\overline{W}_s + \mathcal{D}_s\,dW_s),\\ \overline{X}_t = \overline{x}_0 + \int_0^t \overline{\mu}(\overline{X}_s)\,ds + \overline{\sigma}(\overline{X}_s)\,d\overline{W}_s, \end{cases}}$$

where $\mathcal{D}_s\,\mathcal{D}_s^{\intercal} + \mathcal{C}_s\,\mathcal{C}_s^{\intercal} = \mathrm{Id}_d$ for any $0 \le s \le T$.

(ii) $\widetilde{\mathbb{P}}$ is the joint probability law of $(X^{\mathcal{C}},\overline{X})$.

A new Wasserstein-type distance

Definition.

Given $\mathbb{P}^{\mu,\sigma}_{x_0}$ and $\mathbb{P}^{\overline{\mu},\overline{\sigma}}_{\overline{x}_0}$,

$$\boxed{\widetilde{\mathcal{W}}^2(\mathbb{P}^{\mu,\sigma}_{x_0};\mathbb{P}^{\overline{\mu},\overline{\sigma}}_{\overline{x}_0}) := \left\{ \inf_{\widetilde{\mathbb{P}}\in\widetilde{\Pi}(\mathbb{P}^{\mu,\sigma}_{x_0};\mathbb{P}^{\overline{\mu},\overline{\sigma}}_{\overline{x}_0})} \int_{\Omega^{\otimes 2}} \int_0^T |\omega_s - \overline{\omega}_s|^2 \, ds \, \widetilde{\mathbb{P}}(d\omega, d\overline{\omega}) \right\}^{\frac{1}{2}}.}$$

Important properties

Proposition

One has

$$\mathcal{W}^2(\mathbb{P}^{\mu,\sigma}_{x_0}, \mathbb{P}^{\overline{\mu},\overline{\sigma}}_{\overline{x}_0}) \leq \widetilde{\mathcal{W}}^2(\mathbb{P}^{\mu,\sigma}_{x_0}, \mathbb{P}^{\overline{\mu},\overline{\sigma}}_{\overline{x}_0}).$$

Proposition

The map $\widetilde{\mathcal{W}}^2$ defines a distance on $\mathbf{P}$.

Proposition

Let $\mathbf{P}_{A,\Lambda}$ be the set of the probability distributions $\mathbb{P}^{\mu,\sigma}_{x_0}$ of pathwise unique strong solutions to SDEs with coefficients s.t.

$$\begin{cases} |\mu(x)| \leq A, \ \ |\sigma(x)| \leq A, \ \forall x \in \mathbb{R}^d, \\ |\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq \Lambda |x - y|, \ \forall x, y \in \mathbb{R}^d, \end{cases}$$

and initial condition x_0 in a given compact subset of $\mathbb{R}^d$.
$\widetilde{\mathcal{W}}^2$ metrizes the weak topology on $\mathbf{P}_{A,\Lambda}$.

A stochastic control problem

Let $\mathbf{C}_d$ denote the space of $d \times d$ correlation matrices .

For every $0 \leq t \leq T$ let $\mathbf{Ad(t, T)}$ denote the set of admissible controls between t and T, that is, the set of $\mathcal{G}$-predictable processes on $[t, T]$ which take values in $\mathbf{C}_d$ and are independent of $\mathcal{G}_t$.

For all $0 \leq t \leq T$ and $(\mathcal{C}_\theta)$ in $\mathbf{Ad}(t, T)$ there exists a pathwise unique strong solution to

$$\boxed{\begin{cases} X_\theta^{\mathcal{C}} = x + \int_t^\theta \mu(X_s^{\mathcal{C}})\, ds + \int_t^\theta \sigma(X_s^{\mathcal{C}})\, (\mathcal{C}_s\, d\overline{W}_s + \mathcal{D}_s\, dW_s), \\ \overline{X}_t = \overline{x} + \int_t^\theta \overline{\mu}(\overline{X}_s)\, ds + \int_t^\theta \overline{\sigma}(\overline{X}_s)\, d\overline{W}_s, \end{cases}}$$

where $\mathcal{D}_s\, \mathcal{D}_s^{\mathsf{T}} + \mathcal{C}_s\, \mathcal{C}_s^{\mathsf{T}} = \mathrm{Id}_d$ for any $t \leq s \leq T$.

Consider the objective function

$$\boxed{\min_{(\mathcal{C}_\theta) \in \mathbf{Ad(0,T)}} \mathbb{E} \int_0^T |X_\theta^{\mathcal{C}}(0, x, \overline{x}) - \overline{X}_\theta(0, \overline{x})|^2 d\theta}$$

Hamilton–Jacobi–Bellman equation

$$\boxed{\begin{cases} \partial_t V(t,x,\overline{x}) + \mathcal{L}V(t,x,\overline{x}) + H(t,x,\overline{x},V) = -|x-\overline{x}|^2, \ 0 \le t < T, \\ V(T,x,\overline{x}) = 0, \end{cases}}$$

where

$$\boxed{\begin{aligned} \mathcal{L}V(t,x,\overline{x}) := {} & \sum_i \mu^i(x)\partial_{x_i} V(t,x,\overline{x}) + \sum_i \overline{\mu}^i(\overline{x})\partial_{\overline{x}_i} V(t,x,\overline{x}) \\ & + \frac{1}{2}\sum_{i,j}(\sigma\sigma^\intercal(x))^{ij}\partial^2_{x_i,x_j} V(t,x,\overline{x}) \\ & + \frac{1}{2}\sum_{i,j}(\overline{\sigma}(\overline{x})\overline{\sigma}(\overline{x})^\intercal)^{ij}\partial^2_{\overline{x}_i,\overline{x}_j} V(t,x,\overline{x}) \end{aligned}}$$

and

$$\boxed{H(t,x,\overline{x},V) := \min_{C\in\mathbf{C}_d} \sum_{i,j}(\sigma(x)C\overline{\sigma}(\overline{x})^\intercal)^{ij}\partial^2_{x_i,\overline{x}_j} V(t,x,\overline{x}).}$$

Smooth solution to HJB

Theorem (with J. Bion-Nadal, Ecole Polytechnique)

Suppose:

(i) The functions μ, $\overline{\mu}$, σ and $\overline{\sigma}$ are in the Hölder space $\mathcal{C}^{1+\alpha}(\mathbb{R}^d)$ with $0 < \alpha \leq 1$.

(ii) The matrix-valued functions $a(x) := \sigma(x)\sigma(x)^\intercal$ and $\overline{a}(x) := \overline{\sigma}(x)\overline{\sigma}(x)^\intercal$ satisfy the strong ellipticity condition

$$\exists\lambda > 0,\ \forall\xi, \overline{\xi}, x \in \mathbb{R}^d,\quad \sum_{i,j} a^{ij}(x)\xi^i\xi^j + \sum_{i,j} \overline{a}^{ij}(x)\overline{\xi}^i\overline{\xi}^j \geq \lambda|(\xi|^2 + |\overline{\xi}|^2)$$

Then there exists a unique solution $V(t, x, \overline{x})$ to the HJB equation in $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d)$ such that $\frac{V(t,x,\overline{x})}{1+|x|^2|+|\overline{x}|^2}$ is in $\mathcal{C}^{1+\frac{\alpha}{2},2+\alpha}([0, T] \times \mathbb{R}^{2d})$

Theorem (with J. Bion-Nadal, Ecole Polytechnique)

There exist a filtered probability space equipped with two independent standard Brownian motions W and $\overline{W}$ and a predictable process $(\mathcal{C}^*, \mathcal{D}^*)$ such that there exists an adapted and pathwise unique solution $(X^*, \overline{X})$ on $[t, T]$ to the system

$$\forall t \leq \theta, \quad \begin{cases} X^*_\theta = x + \int_t^\theta \mu(X^*_s)\ ds + \int_t^\theta \ \sigma(X^*_s)\ \mathcal{C}^*_s\ d\overline{W}_s \\ \qquad\qquad + \int_t^\theta \ \sigma(X^*_s)\ \mathcal{D}^*_s\ dW_s, \\ \overline{X}_\theta = x + \int_t^\theta \overline{\mu}(\overline{X}_s)\ ds + \int_t^\theta \overline{\sigma}(\overline{X}_s)\ d\overline{W}_s, \\ \mathcal{C}^*_s \in \arg\min\limits_{C \in \mathbf{O}_d} \sum_{i,j=1}^d \Big(\sigma(X^*_s)\ C\ \overline{\sigma}(\overline{X})^{\intercal}_s)^{ij} \partial^2_{x_i \overline{x}_j} V(s, X^*_s, \overline{X}_s)\Big), \\ \mathcal{C}^*_s\ (\mathcal{C}^*_s)^{\intercal} + \mathcal{D}^*_s\ (\mathcal{D}^*_s)^{\intercal} = \mathrm{Id}_d, \end{cases}$$

which satisfies

$$\boxed{V(0, x_0, x_0) = \mathbb{E} \int_0^T |X^*_\theta(t, x) - \overline{X}_\theta(t, \overline{x})|^2 d\theta = \widetilde{\mathcal{W}}^2(\mathbb{P}^{\mu,\sigma}; \mathbb{P}^{\overline{\mu},\overline{\sigma}})}$$

Outline

1. Sensitivity w.r.t. Hurst parameter of functionals of diffusions driven by fractional Brownian motions
2. On a new Wasserstein type distance
3. Conclusion

Quand je pense à Nicole, les deux mots qui me viennent spontanément à l'esprit sont enthousiasme et générosité.

Enthousiasme pour les probabilités, les mathématiques financières, le contrôle stochastique, les enseignements, les congrès, la musique.

Générosité sans borne pour les étudiants en perdition, les doctorants en période de doute, et les collègues qui sollicitent son énergie ou son savoir.

NICOLE, NO MODELLING ISSUE WITH YOU:
YOU ARE AN EXCELLENT MODEL!