Preserving and propagating convex order: a major asset for numerical schemes (not only) in Finance

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Outline

1. Convex order for (path-dependent) European options
   - From discrete time ARCH and Brownian diffusions
     - From discrete time...
     - ...to continuous time


3. Application to MOT (with B. Jourdain, 2019)
Definition (Convex order, peacock)

(a) Two $\mathbb{R}^d$-valued random vectors $U, V$ are ordered in convex order, denoted

$$U \leq_{cx} V$$

if

$$\forall \varphi : \mathbb{R}^d \to \mathbb{R}, \text{ convex}, \quad \mathbb{E} \varphi(U) \leq \mathbb{E} \varphi(V).$$

(b) A stochastic process $(X_u)_{u \geq 0}$ defined on a probability space is a p.c.o.c. if

$$u \mapsto X_u$$

is non-decreasing for the convex order.

- In particular if $U$ and $V$ are integrable, rthen $\mathbb{E} U = \mathbb{E} V$
- If $(X_t)_t$ is a martingale, then $(X_t)_t$ is a p.c.o.c.
Kellerer’s Theorem: “$X$ is a peacock $\iff X$ is a 1-martingale”.

There exists a martingale $(M_t)_{t \geq 0}$ such that $X_t \overset{d}{=} M_t$, $t \geq 0$.

The proof is unfortunately non-constructive.

Examples

- $\sigma \mapsto e^{\sigma W_t - \frac{\sigma^2 t}{2}}$ is a p.c.o.c. ($\iff$ the vega of a call option is non-negative in a BS model).
- $\sigma \mapsto \frac{1}{T} \int_0^T e^{\sigma W_t - \frac{\sigma^2 t}{2}} \, dt$ is a p.c.o.c. ($\iff$ the vega of a Asian call option is non-negative in a BS model).

Hirsch, Roynette, Profeta & Yor wrote a monography many explicit “representations” of p.c.o.c. by 1-martingales with many extensions...
This suggests many other (new or not so new) questions!

- Monotone convex order: [Hajek, 1985].

- Switch from BS to Local volatility models i.e. $\sigma = \sigma(x)$ (""functional"" convex order)? [El Karoui-Jeanblanc-Schreve, 1998], [Martini, 1999].


- American & Bermuda options? [Pham 2005], [Rüschendorf, 2008].

- Jumpy risky asset dynamics for $(X_t^\sigma)$? [Rüschendorf-Bergenthum, 2007].

Our aims

1. Generalize and unify these results with a focus on path-dependent payoffs (like Asian options) i.e. functional convex order.

2. Constraint: provide a constructive method of proof
   - based on time discretization of continuous time (risky asset) martingale dynamics models
   - using numerical schemes that preserve the functional convex order satisfied by the underlying process...
   - to avoid arbitrages.

3. Apply the paradigm to various frameworks (American style options, BSDEs, jump models, stochastic integrals, etc).
Theorem (P. 2016, Sém. Prob.)

Let $\sigma, \theta \in C_{lin_x}([0, T] \times \mathbb{R}, \mathbb{R})$. Let $X^{(\sigma)}$ and $X^{(\theta)}$ be the unique weak solutions to

$$
\begin{align*}
\frac{dX_t^{(\sigma)}}{dt} &= \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)}, \quad X_0^{(\sigma)} = x \\
\frac{dX_t^{(\theta)}}{dt} &= \theta(t, X_t^{(\theta)})dW_t^{(\theta)}, \quad X_0^{(\theta)} = x,
\end{align*}
$$

$(W_t^{(\cdot)})_{t \in [0, T]}$ standard B.M.

(a) ▷ If there exists a partitioning function $\kappa \in C_{lin_x}([0, T] \times \mathbb{R}, \mathbb{R})$ s.t.

$$
\begin{align*}
(i) & \quad \kappa(t, .) \text{ is convex for every } t \in [0, T], \\
(ii) & \quad 0 \leq \sigma \leq \kappa \leq \theta.
\end{align*}
$$

▷ Then, for every $F : C([0, T], \mathbb{R}) \to \mathbb{R}$, convex, with $\|\cdot\|_{sup}$-polynomial growth (hence $\|\cdot\|_{sup}$-continuous). Then

$$
\mathbb{E} F(X^{(\sigma)}) \leq \mathbb{E} F(X^{(\theta)}).
$$

(b) Domination: If $|\sigma| \leq \theta = \kappa$ (hence convex), the conclusion still holds true.
Illustrations (of Claim (a))

Figure: Left: Convex partitioning. Right: Convex bounding.

Remarks. • When $F(x) = f(x(T))$ a PDE argument (maximum principle) yields the conclusion.
Proposition (A comparison theorem)

If all \( \sigma_k \) or all \( \theta_k \) are convex with linear growth and

\[
\forall, k \in \{0, \ldots, n-1\}, \quad \sigma_k \leq \theta_k,
\]

then

\[
\forall k \in \{0, \ldots, n\}, \quad (X_0, \ldots, X_n) \leq_{cvx} (Y_0, \ldots, Y_n).
\]

> Dynamics: \((Z_k)_{1 \leq k \leq n}\) be a sequence of independent, centered r.v.

\[
X_{k+1} = X_k + \sigma_k(X_k) Z_{k+1},
\]
\[
Y_{k+1} = Y_k + \theta_k(Y_k) Z_{k+1}, \quad k = 0 : n-1, \quad X_0 = Y_0 = x
\]

where \( \sigma_k, \theta_k : \mathbb{R} \to \mathbb{R}, \quad k = 0 : n-1 \) have linear growth.
**Dynamic programming:** We introduce two martingales

\[ M_k = \mathbb{E}(F(X_{0:n}) | \mathcal{F}_k^Z) \quad \text{and} \quad N_k = \mathbb{E}(F(Y_{0:n}) | \mathcal{F}_k^Z), \quad k = 0 : n \]

and the sequence of operators

\[ Q_k(\varphi)(u) = \mathbb{E} \varphi(uZ_k), \quad u \in \mathbb{R}, \quad k = 1 : n. \]
Lemma (Jensen’s Inequality revisited)

Let \( Z : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R} \) be an (integrable) centered r.v.

- Let \( \varphi : \mathbb{R} \to \mathbb{R} \), such that
  \[
  \forall u \in \mathbb{R}^d, \quad Q\varphi(u) := \mathbb{E} \varphi(u Z) \text{ is well defined in } \mathbb{R}.
  \]

If \( \varphi \) is convex, then \( Q\varphi \) is convex, attains its minimum at 0 so that \( Q\varphi \) is non-decreasing on \( \mathbb{R}_+ \), non-increasing on \( \mathbb{R}_- \).

- If \( Z \) has a symmetric distribution, then \( Q\varphi \) is an even function and
  \[
  \forall a \in \mathbb{R}_+, \quad \sup_{|u| \leq a} Q\varphi(u) = Q\varphi(a).
  \]

Proof. The function \( Q\varphi \) is clearly convex and by Jensen’s Inequality
\[
Q\varphi(u) \geq \varphi(\mathbb{E}(u Z)) = \varphi(u \mathbb{E}Z) = \varphi(0) = Q\varphi(0).
\]

Hence \( Q\varphi \) is convex, attains its minimum at \( u = 0 \) hence is non-increasing on \( \mathbb{R}_- \) and non-decreasing on \( \mathbb{R}_+ \). □
Lemma (Jensen’s Inequality revisited)

Let $Z : (\Omega, \mathcal{A}, P) \to \mathbb{R}$ be an (integrable) centered r.v.

- Let $\varphi : \mathbb{R} \to \mathbb{R}$, such that
  \[ \forall u \in \mathbb{R}^d, \quad Q \varphi(u) := E \varphi(u Z) \] is well defined in $\mathbb{R}$.

If $\varphi$ is convex, then $Q \varphi$ is convex, attains its minimum at $0$ so that $Q \varphi$ is non-decreasing on $\mathbb{R}_+$, non-increasing on $\mathbb{R}_-$.

- If $Z$ has a symmetric distribution, then $Q \varphi$ is an even function and
  \[ \forall a \in \mathbb{R}_+, \quad \sup_{|u| \leq a} Q \varphi(u) = Q \varphi(a). \]

Proof. The function $Q \varphi$ is clearly convex and by Jensen’s Inequality

\[ Q \varphi(u) \geq \varphi(E(u Z)) = \varphi(u E Z) = \varphi(0) = Q \varphi(0). \]

Hence $Q \varphi$ is convex, attains its minimum at $u = 0$ hence is non-increasing on $\mathbb{R}_-$ and non-decreasing on $\mathbb{R}_+$. \hfill \square
A (first) **backward induction** and the definition of the kernels $Q_k$ imply

$$M_k = \Phi_k(x_{0:k}) \quad \text{and} \quad N_k = \Psi_k(y_{0:k}), \quad k = 0, \ldots, n.$$  

where $\Phi_k, \Psi_k : \mathbb{R}^{k+1} \to \mathbb{R}, \quad k = 0, \ldots, n$ are recursively defined by

$$\Phi_n := F, \quad \Phi_k(x_{0:k}) := [\mathbb{E} \Phi_{k+1}(x_{0:k}, x_k + uZ_{k+1})]|_{u=\sigma_k(x_k)}$$

$$:= (Q_{k+1}\Phi_{k+1}(x_{0:k}, x_k + .))(\sigma_k(x_k)), \quad k = 0 : n - 1.$$  

Likewise

$$\Psi_n := F, \quad \Psi_k(y_{0:k}) := (Q_{k+1}\Psi_{k+1}(y_{0:k}, y_k + .))(\theta_k(y_k)), \quad k = 0 : n - 1.$$
Assume now that all functions $\sigma_k$ are $\geq 0$ and convex:
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- One has
  \[
  \left( G : \mathbb{R}^{k+2} \to \mathbb{R} \text{ convex} \right)
  \]
  \[
  \left( (x_0:k, u) \mapsto (Q_{k+1} G(x_0:k, x_k+))(u) = \mathbb{E} G(x_0:k, x_k+uZ) \right) \text{ is convex} \ldots
  \]
  
  so that, by the revisited Jensen's Lemma,

  (i) $u \mapsto (Q_{k+1} G(x_0:k, x_k+))(u)$ is ↓ on $(-\infty, 0)$ and ↑ on $(0, +\infty)$.

  &

  (ii) Propagation of the convexity in $x_0:k$. 

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  \]
  so that, by the revisited Jensen’s Lemma,
  \[(i) \quad u \mapsto (Q_{k+1} G(x_{0:k}, x_k+))(u) \text{ is } \downarrow \text{ on } (-\infty, 0) \text{ and } \uparrow \text{ on } (0, +\infty).
  \]
  &
  \[(ii) \text{ Propagation of the convexity in } x_{0:k}.
  \]
- (Second) backward induction $\implies$ all functions $\Phi_k$ are convex.
Assume now that all functions $\sigma_k$ are $\geq 0$ and convex:

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  &

  (ii) Propagation of the convexity in $x_{0:k}$.

- (Second) backward induction $\Longrightarrow$ all functions $\Phi_k$ are convex.

- (Third) backward induction $\Longrightarrow \Phi_k \leq \Psi_k$, $k = 0 : n - 1$.

First note that $\Phi_n = \Psi_n = F$. If $\Phi_{k+1} \leq \Psi_{k+1}$, then

\[
\Phi_k(x_{0:k}) = (Q_{k+1}\Phi_{k+1}(x_{0,k}, x_k+.))(\sigma_k(x_k)) \\
\leq (Q_{k+1}\Phi_{k+1}(x_{0:k}, x_k+.))(\theta_k(x_k)) \\
\leq (Q_{k+1}\Psi_{k+1}(x_{0:k}, x_k+.))(\theta_k(x_k)) = \Psi_k(x_{0:k}).
\]
Assume now that all functions $\sigma_k$ are $\geq 0$ and convex:

- One has

\[
\left( G : \mathbb{R}^{k+2} \to \mathbb{R} \text{ convex} \right) \downarrow \\
\left( (x_{0:k}, u) \mapsto (Q_{k+1} G(x_{0:k}, x_k + .))(u) = \mathbb{E} G(x_{0:k}, x_k + uZ) \right) \text{ is convex.} \
\]

so that, by the revisited Jensen’s Lemma,

(i) $u \mapsto (Q_{k+1} G(x_{0:k}, x_k + .))(u)$ is ↓ on $(-\infty, 0)$ and ↑ on $(0, +\infty)$.

&

(ii) Propagation of the convexity in $x_{0:k}$.

- (Second) backward induction $\implies$ all functions $\Phi_k$ are convex.

- (Third) backward induction $\implies$ $\Phi_k \leq \Psi_k$, $k = 0 : n - 1$.

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\Phi_k(x_{0:k}) = (Q_{k+1} \Phi_{k+1}(x_{0:k}, x_k + .))(\sigma_k(x_k)) \\
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\leq (Q_{k+1} \Psi_{k+1}(x_{0:k}, x_k + .))(\theta_k(x_k)) = \Psi_k(x_{0:k}).
\]
If all $\theta_k \geq 0$ and convex:

This time, one shows that:

- the functions $\Psi_k$ are convex, $k = 0, \ldots, n$
- $\Phi_n \leq \Psi_n \implies \Phi_k \leq \Psi_k$, $k = 0, \ldots, n - 1$.

**Remark.** The discrete time setting has its own interest.
Step 2 of the proof: Back to continuous time
Step 2 of the proof: Back to continuous time

▷ Euler scheme(s): Discrete time Euler scheme with step $\frac{T}{n}$, starting at $x$ is an ARCH model. For $X^{(\sigma)}$: for $k = 0, \ldots, n - 1$,

\[
\bar{X}^{(\sigma),n}_{t_{k+1}} = \bar{X}^{(\sigma),n}_{t_k} + \sigma(t_k, \bar{X}^{(\sigma),n}_{t_k}) \left( W_{t_{k+1}} - W_{t_k} \right), \quad \bar{X}^{(\sigma),n}_{0} = x
\]

Set

\[
Z_k = W_{t_k^n} - W_{t_{k-1}^n}, \quad k = 1, \ldots, n
\]

\[
\downarrow
\]

discrete time setting applies

Remark. Linear growth of $\sigma$ and $\theta$, implies

\[
\forall p > 0, \quad \sup_{n \geq 1} \left\| \sup_{t \in [0, T]} \left| \bar{X}^{(\sigma),n}_t \right| \right\|_p + \sup_{n \geq 1} \left\| \sup_{t \in [0, T]} \left| \bar{X}^{(\theta),n}_t \right| \right\|_p < +\infty.
\]
From discrete to continuous time

Interpolation \((n \geq 1)\)

- **Piecewise affine interpolator** defined by

\[
\forall x_{0:n} \in \mathbb{R}^{n+1}, \, \forall k = 0, \ldots, n-1, \, \forall t \in [t^n_k, t^n_{k+1}],
\]

\[
i_n(x_{0:n})(t) = \frac{n}{T}((t^n_{k+1} - t)x_k + (t - t^n_k)x_{k+1})
\]

- \(\tilde{X}^{(\sigma),n} := i_n(\tilde{X}^{(\sigma),n}_{t^n_k})_{k=0:n} = \) piecewise affine Euler scheme.
\( F : C([0, T], \mathbb{R}) \to \mathbb{R} \) convex functional (with \( r \)-polynomial growth).

\[
\forall n \geq 1, \quad F_n(x_{0:n}) := F(i_n(x_{0:n})), \quad x_{0:n} \in \mathbb{R}^{n+1}.
\]
\[ F : \mathcal{C}([0, T], \mathbb{R}) \to \mathbb{R} \text{ convex functional (with } r\text{-polynomial growth)}. \]

\[ \forall n \geq 1, \quad F_n(x_0:n) := F(i_n(x_0:n)), \quad x_0:n \in \mathbb{R}^{n+1}. \]

\textbf{Step 1 (Discrete time)}:

\[ F \text{ convex } \implies F_n \text{ convex, } n \geq 1. \]

Discrete time result implies:

\[ \left[ \sigma(t^n_k, .) \leq \kappa(t^n_k, .) \leq \theta(t^n_k, .) \right]. \]

\[ \mathbb{E} F(\tilde{X}^{(\sigma), n}) = \mathbb{E} F_n((\tilde{X}_{t^n_k}^{(\sigma), n})_{k=0:n}) \leq \mathbb{E} F_n((\tilde{X}_{t^n_k}^{(\kappa), n})_{k=0:n}) \leq \mathbb{E} F_n((\tilde{X}_{t^n_k}^{(\theta), n})_{k=0:n}) = \mathbb{E} F(\tilde{X}^{(\theta), n}). \]
\[ F : \mathcal{C}([0, T], \mathbb{R}) \rightarrow \mathbb{R} \text{ convex functional (with } r\text{-polynomial growth).} \]

\[ \forall n \geq 1, \quad F_n(x_{0:n}) := F\left(\inf_{x_{0:n}}(x_{0:n})\right), \quad x_{0:n} \in \mathbb{R}^{n+1}. \]

**Step 1 (Discrete time):**

\[ F \text{ convex } \implies F_n \text{ convex, } n \geq 1. \]

Discrete time result implies: \[ \left[ \sigma(t^n_k, .) \leq \kappa(t^n_k, .) \leq \theta(t^n_k, .) \right]. \]

\[ \mathbb{E} F\left(\tilde{X}^{(\sigma), n}\right) = \mathbb{E} F_n\left((\tilde{X}_{t^n_k}^{(\sigma), n})_{k=0:n}\right) \leq \mathbb{E} F_n\left((\tilde{X}_{t^n_k}^{(\kappa), n})_{k=0:n}\right) \leq \mathbb{E} F_n\left((\tilde{X}_{t^n_k}^{(\theta), n})_{k=0:n}\right) = \mathbb{E} F\left(\tilde{X}^{(\theta), n}\right). \]

**Step 2 (Transfer):** Key = Theorem 3.39, p.551, Jacod-Shiryaev’s book (2\textsuperscript{nd} edition).

\[ \tilde{X}^{(\sigma), n} \xrightarrow{\mathcal{L}(\|\cdot\|_{\sup})} X^{(\sigma)} \quad \text{as } n \rightarrow \infty. \]

\[ \mathbb{E} F\left(X^{(\sigma)}\right) = \lim_{n} \mathbb{E} F\left(\tilde{X}^{(\sigma), n}\right) \quad (\text{idem for } X^{(\theta)}). \]
The **Euler scheme** is a simulable approximation which preserves convex order.
Application I : Local Volatility models (functional peacocks).

- New notations ($\sigma$, $\theta$ are now true \textit{volatility})

\[dS_t = S_t \tilde{\sigma}(t, S_t) dW_t, \quad S_0 = s_0 > 0, \quad (r = 0)\]

where $\tilde{\sigma} : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a \textbf{bounded continuous function}.

- The weak solution satisfies

\[S_t^{(\tilde{\sigma})} = s_0 e^{\int_0^t \tilde{\sigma}(s, S_s^{(\sigma)}) dB_s - \frac{1}{2} \int_0^t \sigma^2(s, S_s^{(\sigma)}) ds} > 0.\]

- Idem for $\theta$.

- We assume that $0 \leq \tilde{\sigma} \leq \tilde{\kappa} \leq \tilde{\theta}$ and $\kappa : x \mapsto x\tilde{\kappa}(t, x)$ is convex on the whole real line.
Convex order for (path-dependent) European options
From discrete time ARCH and Brownian diffusions

Example of application: El Karoui-Jeanblanc-Shreve’s Theorem.

Theorem (Extension of El Karoui et al. theorem., P. 2016)

If there exists a function \( \tilde{\kappa} : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \kappa : x \mapsto x\tilde{\kappa}(t, x) \) extends into an \( \mathbb{R}_+ \)-valued convex function on \( \mathbb{R} \) satisfying

(a) **Partitioning:** \( 0 \leq \tilde{\sigma}(t, \cdot) \leq \tilde{\kappa}(t, \cdot) \leq \tilde{\theta}(t, \cdot) \) on \( \mathbb{R}_+ \), \( t \in [0, T] \),
or

(b) **Dominating:** \( |\tilde{\sigma}(t, \cdot)| \leq \tilde{\theta}(t, \cdot) = \tilde{\kappa}(t, \cdot) \) \( t \in [0, T] \).

If \( f : \mathbb{R} \to \mathbb{R} \) convex function and \( \mu \) is a (finite) signed measure on \( [0, T] \)

\[
\mathbb{E} f \left( \int_0^T S^{(\tilde{\sigma})}_s \mu(ds) \right) \leq \mathbb{E} f \left( \int_0^T S^{(\tilde{\theta})}_s \mu(ds) \right) \in (-\infty, +\infty].
\]

and more generally, for every every functional \( F \) with \( r \)-polynomial growth

\[
\mathbb{E} F \left( S^{(\tilde{\sigma})} \right) \leq \mathbb{E} F \left( S^{(\tilde{\theta})} \right) \in \mathbb{R}.
\]
The method of proof applies to American style options, Lévy driven diffusions, stochastic integrals, etc.

1D-Misltein scheme.
Convex Local Volatility (CLV) models (⊇ CEV):
Set \( \sigma(x) = x \tilde{\sigma}(x) > 0 \) on \((0, +\infty)\), \( \sigma(x) = 0, \ x \leq 0 \).

\[
dS_t = \sigma(S_t) dW_t, \ S_0 = s_0 > 0, \ \sigma \text{ concave and } > 0
\]
such that the (unique weak) solution satisfies \( S_t \geq 0, \ t \in [0, T] \).

Then, for every fixed \( u > 0 \), the concavity property implies

\[
\sigma(x) \leq \left( \sigma(u) + \sigma'(u)(x - u) \right)_+, \ x \in \mathbb{R}
\]
so that, if we set

\[
dX_t^{(u)} = \left( \sigma(u) + \sigma'(u)(X_t^{(u)} - u) \right)_+ dW_t, \ X_0^{(u)} = s_0
\]
then, for every convex vanilla payoff \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \)

\[
\mathbb{E} \varphi(S_T) \leq \inf_{u > 0} \mathbb{E} \varphi(X_T^{(u)}).
\]
**Figure**: Black-Scholes convex domination of a Local Vol. model
Set $\theta(u) := \frac{\sigma'(u)}{\sigma(u)} - u > 0$ (by concavity). Hence

$$X_t^{(u)} + \theta(u) = s_0 + \theta(u) + \int_0^t \sigma'(u)(X_s^{(u)} + \theta(u))_+ dW_s > 0$$

i.e., $Y_t^{(u)} = X_t^{(u)} + \theta(u)$, satisfies the Black-Scholes dynamics

$$Y_t^{(u)} = Y_0^{(u)} + \sigma'(u) \int_0^t Y_s^{(u)} dW_s.$$ 

**Example:** if $\varphi(x) = (x - K)_+$ is a vanilla Call payoff

$$\mathbb{E}(S_T - K)_+ \leq \inf_{u > 0} \text{Call}_{BS} \left( s_0 + \theta(u), K + \theta(u), \sigma'(u) \right)$$
Proposition (Tractable upper-bound)

One has

(i) \( u \mapsto \mathbb{E}(Y_T^{(u)} - K)_+ \) is differentiable and \( \frac{\partial}{\partial u} \mathbb{E}(Y_T^{(u)} - K)_+ \geq 0 \) on \( \left[ \max(s_0, K), +\infty \right) \)

(ii) Hence

\[
\mathbb{E}(S_T - K)_+ \leq \min_{0 \leq u \leq \max(s_0, K)} \text{Call}_{BS}(s_0 + \theta(u), K + \theta(u), \sigma'(u))
\]

leading to a faster **search** for the argmin.

Practitioner’s corner: – In fact \( u_{\min} \) lies not far from \( s_0 \) and \( K \).

– Exploration starting from \( \frac{s_0 + K}{2} \).
When the drift comes back into the game: **non-decreasing convex order** for diffusions

**Theorem (Extended Hajek’s Theorem, P. 2016, Sém. Prob. XLVIII,)**

Let $\sigma, \theta \in C_{lin_x}([0, T] \times \mathbb{R}, \mathbb{R})$. Let $X^{(\sigma)}$ and $X^{(\theta)}$ be the unique weak solutions to

\[
\begin{align*}
    dX_t^{(\sigma)} &= b(t, X_t^{(\sigma)}) dt + \sigma(t, X_t^{(\sigma)}) dW_t^{(\sigma)}, \quad X_0^{(\sigma)} = x \\
    dX_t^{(\theta)} &= b(t, X_t^{(\theta)}) dt + \theta(t, X_t^{(\theta)}) dW_t^{(\theta)}, \quad X_0^{(\theta)} = x,
\end{align*}
\]

with $(W_t^{(\cdot)})_{t \in [0, T]}$ is a standard dim B.M. Then,

(i) $b(t, .)$ convex $t \in [0, T]$ and $|b(t, x)| \leq C(1 + |x|)$.
(ii) $\kappa$-partitioning or dominating assumption.

\[\triangleright\] Then, for every $f : \mathbb{R} \to \mathbb{R}$, convex and non-decreasing, with polynomial growth,

\[\mathbb{E} f(X_T^{(\sigma)}) \leq \mathbb{E} f(X_T^{(\theta)}).\]
Multidimensional extension (with A. Fadili, 2017)

- Pre-order $\preceq$ on $\mathcal{M}(d, q, \mathbb{R})$: let $A, B \in \mathcal{M}(d, q, \mathbb{R})$.
  \[ A \preceq B \quad \text{if} \quad AA^* \preceq BB^* \quad \text{in} \quad S(d, \mathbb{R}). \]

- $\preceq$-Convexity: A function $\phi: \mathbb{R}^d \to \mathcal{M}(d, q, \mathbb{R})$ is $\preceq$-convex if: for every $x, y \in \mathbb{R}^d$, and every $\lambda \in [0, 1]$,
  \[ \phi(\lambda x + (1 - \lambda)y) \preceq \lambda \phi(x) + (1 - \lambda)\phi(y). \]

**Proposition**

Let $A, B \in \mathcal{M}(d, q, \mathbb{R})$ such that $A \preceq B$. Let $Z \sim \mathcal{N}(0, I_q)$. Then, for every $\preceq$-convex function $f: \mathbb{R}^d \to \mathbb{R}$,

\[ \mathbb{E} f(AZ) \leq \mathbb{E} f(BZ). \]

- The former theorem formally extended remains valid with these definitions for diffusion models of the form
  \[ dX_t = \sigma(X_t) dW_t, \quad \sigma: \mathbb{R}^d \to \mathcal{M}(d, q, \mathbb{R}), \quad Z \sim \mathcal{N}(0, I_q). \]
**Theorem (with A. Fadili, 2017)**

Let $\sigma, \theta \in C_{\text{lin}_x}([0, T] \times \mathbb{R}, \mathcal{M}(d, q, \mathbb{R}))$, $W^{(\sigma)}$, $W^{(\theta)}$ $q$-S.B.M.. Let $X^{(\sigma)}$ and $X^{(\theta)}$ be the unique weak solutions to

$$
\begin{align*}
    dX^{(\sigma)}_t &= \sigma(t, X^{(\sigma)}_t)\,dW^{(\sigma)}_t, \quad X^{(\sigma)}_0 = x, \\
    dX^{(\theta)}_t &= \theta(t, X^{(\theta)}_t)\,dW^{(\theta)}_t, \quad X^{(\theta)}_0 = x, \\
    (W^{(\cdot)}_t)_{t \in [0, T]} &= \text{standard B.M.}
\end{align*}
$$

(a) ▷ If there exists a partitioning function $\kappa \in C_{\text{lin}_x}([0, T] \times \mathbb{R}, \mathcal{M}(d, q, \mathbb{R}))$ s.t.

$$
\begin{align*}
    &\{ (i) \quad \kappa(t, .) : \mathbb{R}^d \to \mathcal{M}(d, q, \mathbb{R}) \text{ is } \preceq\text{-convex for every } t \in [0, T], \\
    &\quad (ii) \quad \sigma \preceq \kappa \preceq \theta. \}
\end{align*}
$$

▷ Then, for every $F : C([0, T], \mathbb{R}^d) \to \mathbb{R}$, convex, with $\| . \|_{\text{sup}}$-polynomial growth (hence $\| . \|_{\text{sup}}$-continuous). Then

$$
\mathbb{E} F(X^{(\sigma)}) \leq \mathbb{E} F(X^{(\theta)}).
$$

(b) **Domination:** If $\sigma \preceq \theta = \kappa$ (hence convex), the conclusion still holds true.
Extensions

This provides as systematic approach which successfully works with

- Jump diffusion models,
- Path-dependent American style options,
- BSDE (without “Z” in the driver),
- ...

G. PAGÈS (LPSM)
Another kind of application: MOT

- Let $X_{0:n}$ be a martingale on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution $\mu \in \mathcal{P}((\mathbb{R}^d)^{n+1})$ and marginal distributions $\mu_k$, $k = 0 : n$.
- Let $c : (\mathbb{R}^d)^{n+1} \to \mathbb{R}_+$ be a cost/payoff function (exotic option...).
- Assume the marginal distributions are fixed (as the result of a calibrations on vanilla options). Solve

$$\text{(MOT)} \equiv \inf \sup \left\{ \mathbb{E}c(X_{0:n}), \ X \text{ martingale, } X_k \sim \mu_k \right\}$$

- Yields bounds on the exotic option premium.
Think of $X = (X_0 : n)$ as an Euler time discretization scheme of a diffusion.

Then

$$X_{k+1} = X_k + \vartheta_k(X_k)Z_{k+1}, \ k = 0 : n - 1, \ Z_k \ i.i.d.$$ 

with $\vartheta_k : \mathbb{R}^d \to \mathbb{M}_{d,q}$, $Z_1 \sim \mathcal{N}(0, I_q)$.

The (MOT) problem cannot be solved as set: it requires

- a space discretization:

$$(\hat{X}_0, \hat{X}_1, \ldots, \hat{X}_n) \simeq (X_0, X_1, \ldots, X_n)$$

where each $\hat{X}_k$ takes finitely many values.

- satisfying monotony for convex order (to avoid arbitrages):

$$\hat{X}_0 \leq_{cvx} \hat{X}_1 \leq_{cvx} \cdots \leq_{cvx} \hat{X}_n,$$

- comparison results for convex order with respect to $X$,
- A complexity kept under control.
A solution: Dual quantization at level $N \geq 1$!

- Let $Y : (\Omega, A, P) \to \mathbb{R}^d$, $Y \sim \mu$, be a random vector lying in $L^\infty(P)$.
- Let $N$ denote a fixed level $N \geq 1$.
- The optimal dual quantization problem introduced by [Wilbertz-P., ’12] reads

$$d_{p,N}(Y) = \inf_{\hat{X}} \left\{ \| Y - \hat{Y} \|_p : \hat{X} : (\Omega \times \Omega_0, A \otimes A_0, P \otimes P_0) \to \mathbb{R}^d, \right.$$  
$$\left. \quad \text{card}\hat{Y}(\Omega \times \Omega_0) \leq N \text{ and } E(\hat{Y} \mid Y) = Y \right\}$$

or, equivalently,

$$d_{p,N}(\mu) = \inf_{X} \left\{ \| Y - V \|_p, (Y, V) : (\Omega, A, P) \to \mathbb{R}^d \times \mathbb{R}^d, \right.$$  
$$\left. \quad Y \sim \mu, E(Y \mid V) = Y, \text{card}(V(\Omega)) \leq N \right\}.$$
Assume \( Y \in L^\infty(\Omega, \mathcal{A}, \mathbb{P}) \) with \( \text{card}(Y(\Omega)) \geq N \). 

(a) Let \( \Gamma \subset \mathbb{R}^d, \Gamma \supset \text{conv}(\text{supp}(\mathcal{L}(Y))) \) with points in general position. Then

\[
\hat{Y}^{\text{del}, \Gamma} = \text{Proj}_{\Gamma}^{\text{del}}(Y, U_{[0,1]})
\]

where \( (D_k(\Gamma))_{1 \leq k \leq m} \) is a Delaunay hyper-triangulation of \( \text{conv}(\Gamma^*, N) \) satisfies the dual stationarity equation

\[
\mathbb{E}(\hat{Y}^{\text{del}, \Gamma} | Y) = Y
\]

and

\[
\| Y - \hat{Y}^{\text{del}, \Gamma} \|_2 = \inf_{\chi} \left\{ \| Y - V \|_p, (Y, V) : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d \times \Gamma, \right. \\
Y \sim \mu, \mathbb{E}(Y | V) = Y, \text{card}(V(\Omega)) \leq N \}.
\]
Theorem

(b) The above infimum in $d_{p,N}(X)$ is always a minimum: there exists a grid $\Gamma^*,N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ of size $N$ whose points are in general position such that

$$d_{p,N}(Y) = \|Y - \hat{Y}^{de},N\|_p \quad \text{with} \quad \hat{Y}^{de},N = \text{Proj}_{\Gamma^*,N}(Y, U_{[0,1]})$$
Application to MOT (with B. Jourdain, 2019)

The Delaunay projection/splitting operator

Figure: Delaunay splitting of $X(\omega)$. 
Dual quantization of \( (B_1, \sup_{t \in [0,1]} B_t) \) (truncated)
Dual quantization of \((B_1, \sup_{t \in [0,1]} B_t)\) (truncated)
The (martingale) dynamics. Set

\[ X_k = F_{k-1}(X_{k-1}, Z_k), \quad (Z_k) \text{ i.i.d;} \sim \mathcal{N}(0; I_q) \]

with \( F_{k-1}(x, z) \mapsto x + \vartheta_{k-1}(x)z \) \( k = 1, \ldots, n - 1 \).

Pre-processing. Discretize the Gaussian white noise by \( \varphi : \mathbb{R}^d \to \Gamma \subset \mathbb{R}^q \),

\[ \tilde{Z}_k = \varphi(Z_k) \text{ s.t. } \mathbb{E} \tilde{Z}_k = 0, \quad Z_k - \tilde{Z}_k \perp \mathbb{L}^2 \tilde{Z}_k, \]

Example: Optimal Voronoi quantization i.e. \( \tilde{Z}_k = \text{Proj}^{\text{vor}}_{\Gamma Z}(Z_k) \), \( k = 1 : n \) which satisfies

\[ \mathbb{E}(Z_k \mid \tilde{Z}_k) = \tilde{Z}_k, \quad k = 1, \ldots, n. \]
The doubly quantized scheme. Define by induction, based on grids $\Gamma_0, \ldots, \Gamma_n$,

(i) $\hat{X}_0 = \text{Proj}^{\text{vor}}_{\Gamma_0}(X_0)$ (Voronoi “nearest neighbour” projection)
The doubly quantized scheme. Define by induction, based on grids $\Gamma_0, \ldots, \Gamma_n$,

(i) $\hat{X}_0 = \text{Proj}_{\Gamma_0}^{\text{vor}}(X_0)$ (Voronoï “nearest neighbour” projection)

(ii) $\tilde{X}_k = F_{k-1}(\hat{X}_{k-1}, \check{Z}_k), \quad k = 1 : n$
The doubly quantized scheme. Define by induction, based on grids $\Gamma_0, \ldots, \Gamma_n$.

(i) $\hat{X}_0 = \text{Proj}^{\text{vor}}_{\Gamma_0}(X_0)$ (Voronoi “nearest neighbour” projection)

(ii) $\tilde{X}_k = F_{k-1}(\hat{X}_{k-1}, \tilde{Z}_k), \quad k = 1 : n$

(iii) $\hat{X}_k = \text{Proj}^{\text{del}}_{\Gamma_k}(\tilde{X}_k, U_k)$

where

$$(U_{1:n}) \text{ is i.i.d. } U([0, 1])-\text{distributed, } \perp \perp (Z_{1:n}), \perp \perp X_0.$$ 

**Proposition (Jourdain-P., ’19)**

(a) The sequence $(\hat{X}_k)_{k=0:n}$ is martingale Markov chain so that

$$\hat{X}_0 \leq_{\text{cvx}} \hat{X}_1 \leq_{\text{cvx}} \cdots \leq_{\text{cvx}} \hat{X}_n.$$ 

(b) Moreover, if the $\theta_k$ are convex (in a matrix sense of $d$ or $q \geq 2$) then

$$\forall k = 0 : n, \quad \hat{X}_k \leq_{\text{cvx}} X_k.$$
A short proof of (a)

- $\hat{X}_0 = \text{Proj}^{\text{vor}}_{\Gamma_0}(X_0)$ is $\Gamma_0$-valued hence has compact support.
- If $\hat{X}_{k-1}$ have compact support, then $\tilde{X}_k$ has compact support since $\tilde{Z}_k$ has. Hence there exists $\Gamma_n \supset \text{conv}(\tilde{X}_k(\Omega))$ and one can define $\hat{X}_k = \text{Proj}^{\text{del}}_{\Gamma_k}(\tilde{X}_k, U_k)$.
- The Markov property is obvious since $\hat{X}_k = \tilde{F}_{k-1}(\tilde{X}_{k-1}, (U_k, Z_k))$.
- Finally, by the universal dual stationarity property

$$
\mathbb{E}(\hat{X}_k \mid \sigma(X_0, Z_\ell, \ell = 1 : k-1, U_\ell, \ell = 1, k-1)) = \left[ \mathbb{E} \text{Proj}^{\text{del}}_{\Gamma_k}(x, U_k) \right]_{x=\tilde{X}_k}
$$

and the martingality of the “kernel” $F_{k-1}$ yields

$$
\mathbb{E}(\tilde{X}_k \mid \mathcal{F}_{k-1}^{X_0, Z, U}) = \hat{X}_{k-1}
$$

so that

$$
\mathbb{E}(\hat{X}_k \mid \mathcal{F}_{k-1}^{X_0, Z, U}) = \hat{X}_{k-1}
$$
Error bound

Theorem (Jourdain-P. '19)

Assume all $\vartheta_k$ are $[\vartheta]_{\text{Lip}}$-Lipschitz continuous and all quantizations are optimal/optimized at respective levels $|\Gamma_k| = N_k$ and $|\Gamma^Z| = N$. For every $k = 0 : n$,

$$\left\| \hat{X}_k - X_k \right\|_2^2 \leq \left( C_{d,\eta}^{\text{vor}} \right)^2 \left( 1 + q[\vartheta]_{\text{Lip}}^2 \right)^k \frac{\sigma_{2+\eta}^2(X_0)}{N_0^{2/d}}$$

$$+ \sum_{\ell=1}^{k} (1 + q[\vartheta]_{\text{Lip}}^2)^{k-\ell} \left[ \left\| \vartheta_{\ell-1}(X_{\ell-1}) \right\|_2^2 \left( C_{q,\eta}^{\text{vor}} \right)^2 \frac{\sigma_{2+\eta}^2(Z)}{N^{2/q}} + \left( \tilde{C}_{d,\eta}^{\text{del}} \right)^2 \frac{\sigma_{2+\eta}^2(\tilde{X}_k)}{N_k^{2/d}} \right].$$