Preserving and propagating convex order: a major asset for numerical schemes (not only) in Finance

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NEK a 3×25 ans

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Outline

1. Convex order for (path-dependent) European options
   - From discrete time ARCH and Brownian diffusions
     - From discrete time. . .
     - . . . to continuous time
2. Multidimensional extension (with A. Fadili, 2017)
3. Application to MOT (with B. Jourdain, 2019)

Definitions

Definition (Convex order, peacock)

(*a*) Two $\mathbb{R}^d$-valued random vectors U, V are ordered in convex order, denoted

$$U \leq_{cx} V$$

if

$$\forall\, \varphi : \mathbb{R}^d \to \mathbb{R}, \text{convex}, \quad \mathbb{E}\,\varphi(U) \leq \mathbb{E}\,\varphi(V).$$

(*b*) A stochastic process $(X_u)_{u\geq 0}$ defined on a probability space is a p.c.o.c. if

$$u \longmapsto X_u \text{ is non-decreasing for the convex order.}$$

- In particular if U and V are integrable, rthen $\mathbb{E}\,U = \mathbb{E}\,V$
- If $(X_t)_t$ is a martingale, then $(X_t)_t$ is a p.c.o.c.

About the converse...

- Kellerer's Theorem: "X is a peacock $\Longleftrightarrow$ X is a 1-martingale".

  There exists a martingale $(M_t)_{t\geq 0}$ such that $X_t \stackrel{d}{=} M_t, \quad t \geq 0.$

  The proof is unfortunately non-constructive.
- **Examples**
  - $\sigma \mapsto e^{\sigma W_t - \frac{\sigma^2 t}{2}}$ is a p.c.o.c. ($\Longrightarrow$ the vega of a call option is non-negative in a BS model).
  - $\sigma \mapsto \frac{1}{T}\int_0^T e^{\sigma W_t - \frac{\sigma^2 t}{2}} dt$ is a p.c.o.c. ($\Longrightarrow$ the vega of a Asian call option is non-negative in a BS model).
- Hirsch, Roynette, Profeta & Yor wrote a monography many explicit "representations" of p.c.o.c. by 1-matingales with many extensions...

A revival motivated by Finance...

▷ This suggests many other (new or not so new) questions !

- Monotone convex order : [Hajek, 1985].
- Switch from *BS* to Local volatility models *i.e* $\sigma = \sigma(x)$ (““functional” convex order) ? [El Karoui-Jeanblanc-Schreve, 1998], [Martini, 1999].
- More general path-dependent payoff functions i.e. “path-dependent” convex order ? [Brown, Rogers, Hobson 2001, Rüschendorf, 2008].
- American & Bermuda options ? [Pham 2005], [Rüschendorf, 2008].
- Jumpy risky asset dynamics for (X_t^σ) ? [Rüschendorf-Bergenthum, 2007].
- Peacocks trough optimal transport. [Beigelbock, Tan, Touzi, Henry-Labordère et al, 2013].

Our aims

1. Generalize and unify these results with of focus on path-dependent payoffs (like Asian options) i.e. functional convex order.
2. Constraint: provide a constructive method of proof
   - based on time discretization of continuous time (risky asset) martingale dynamics models
   - using numerical schemes that preserve the functional convex order satisfied by the underlying process...
   - to avoid arbitrages.
3. Apply the paradigm to various frameworks (American style options, BSDEs, jump models, stochastic integrals, etc).

Path-dependent European options & Brownian diffusions

Theorem (P. 2016, *Sém. Prob.*)

Let $\sigma, \theta \in \mathcal{C}_{lin_x}([0,T]\times\mathbb{R},\mathbb{R})$. *Let* $X^{(\sigma)}$ *and* $X^{(\theta)}$ *be the* unique *weak solutions to*

$$\begin{aligned} dX_t^{(\sigma)} &= \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},\; X_0^{(\sigma)} = x \\ dX_t^{(\theta)} &= \theta(t, X_t^{(\theta)})dW_t^{(\theta)},\; X_0^{(\theta)} = x, \quad (W_t^{(\cdot)})_{t\in[0,T]} \text{ standard B.M.} \end{aligned}$$

(a) ▷ *If there exists a partitioning function* $\kappa \in \mathcal{C}_{lin_x}([0,T]\times\mathbb{R},\mathbb{R})$ *s.t.*

$$\begin{cases} (i) & \kappa(t,.) \text{ is convex for every } t\in[0,T], \\ (ii) & 0\le\sigma\le\kappa\le\theta. \end{cases}$$

▷ *Then, for every* $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$, *convex, with* $\|\,.\,\|_{\sup}$*-polynomial growth (hence* $\|\,.\,\|_{\sup}$*-continuous). Then*

$$\mathbb{E}\,F(X^{(\sigma)}) \le \mathbb{E}\,F(X^{(\theta)}).$$

(b) *Domination: If* $|\sigma|\le\theta=\kappa$ *(hence convex), the conclusion still holds true.*

Illustrations (of Claim (*a*))

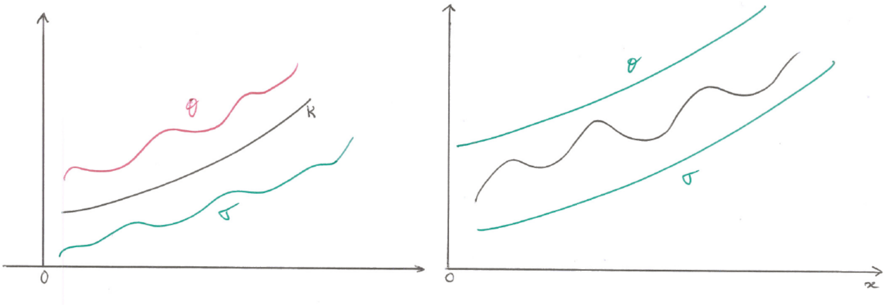

Figure: Left: Convex partitioning. Right: Convex bounding.

Remarks. • When $F(x) = f\big(x(T)\big)$ a PDE argument (maximum principle) yields the conclusion.

Step 1: discrete time ARCH models

Proposition (A comparison theorem)

If all σ_k or all θ_k are convex with linear growth and

$$\forall, k \in \{0, \ldots, n-1\}, \quad \sigma_k \leq \theta_k,$$

then

$$\forall\, k \in \{0, \ldots, n\}, \quad (X_0, \ldots, X_n) \leq_{cvx} (Y_0, \ldots, Y_n).$$

▷ Dynamics: $(Z_k)_{1 \leq k \leq n}$ be a sequence of independent, centered r.v.

$$\begin{aligned} X_{k+1} &= X_k + \sigma_k(X_k)\, Z_{k+1}, \\ Y_{k+1} &= Y_k + \theta_k(Y_k)\, Z_{k+1}, \quad k = 0 : n-1,\ X_0 = Y_0 = x \end{aligned}$$

where σ_k, $\theta_k : \mathbb{R} \to \mathbb{R}$, $k = 0 : n-1$ have linear growth.

$\triangleright$ Dynamic programming: We introduce two martingales

$$M_k = \mathbb{E}\big(F(X_{0:n})\,|\,\mathcal{F}_k^Z\big) \;\text{ and }\; N_k = \mathbb{E}\big(F(Y_{0:n})\,|\,\mathcal{F}_k^Z\big),\; k = 0:n$$

and the sequence of operators

$$Q_k(\varphi)(u) = \mathbb{E}\,\varphi(uZ_k),\; u \in \mathbb{R},\; k = 1:n.$$

Jensen's Inequality (not really) revisited = Key Lemma

Lemma (Jensen's Inequality revisited)

Let $Z : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ be an (integrable) centered r.v.

▷ *Let $\varphi : \mathbb{R} \to \mathbb{R}$, such that*

$$\forall\, u \in \mathbb{R}^d,\ Q\varphi(u) := \mathbb{E}\,\varphi\big(u\,Z\big) \text{ is well defined in } \mathbb{R}.$$

If φ is convex, then $Q\varphi$ is convex, attains its minimum at 0 so that

$Q\varphi$ is non-decreasing on $\mathbb{R}_+$, non-increasing on $\mathbb{R}_-$.

▷ *If Z has a symmetric distribution, then $Q\varphi$ is an even function and*

$$\forall\, a \in \mathbb{R}_+, \quad \sup_{|u|\le a} Q\varphi(u) = Q\varphi(a).$$

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$$\forall\, a \in \mathbb{R}_+, \quad \sup_{|u|\le a} Q\varphi(u) = Q\varphi(a).$$

Proof. The function $Q\varphi$ is clearly convex and by Jensen's Inequality

$$Q\varphi(u) \ge \varphi\big(\mathbb{E}(u\,Z)\big) = \varphi\big(u\,\mathbb{E}\,Z\big) = \varphi(0) = Q\varphi(0).$$

Hence $Q\,\varphi$ is convex, attains its minimum at $u = 0$ hence is non-increasing on $\mathbb{R}_-$ and non-decreasing on $\mathbb{R}_+$. $\square$

- A (first) backward induction and the definition of the kernels Q_k imply

$$M_k = \Phi_k(X_{0:k}) \quad \text{and} \quad N_k = \Psi_k(Y_{0:k}), \quad k = 0, \ldots, n.$$

where $\Phi_k, \Psi_k : \mathbb{R}^{k+1} \to \mathbb{R}$, $k = 0, \ldots, n$ are recursively defined by

$$\begin{aligned}\Phi_n := F, \;\; \Phi_k(x_{0:k}) &= \big[\mathbb{E}\,\Phi_{k+1}(x_{0:k}, x_k + uZ_{k+1})\big]_{|u=\sigma_k(x_k)} \\ &:= \big(Q_{k+1}\Phi_{k+1}(x_{0:k}, x_k + .)\big)(\sigma_k(x_k)),\; k = 0 : n-1.\end{aligned}$$

Likewise

$$\Psi_n := F, \;\; \Psi_k(y_{0:k}) := \big(Q_{k+1}\Psi_{k+1}(y_{0:k}, y_k + .)\big)(\theta_k(y_k)),\; k = 0 : n-1.$$

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$$\Big(G : \mathbb{R}^{k+2} \to \mathbb{R} \text{ convex}\Big)$$
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$$\Big((x_{0:k}, u) \mapsto (Q_{k+1} G(x_{0:k}, x_k + .))(u) = \mathbb{E}\, G(x_{0:k}, x_k + uZ)) \text{ is convex}\dots\Big)$$

so that, by the revisited Jensen's Lemma,

(i) $\quad u \mapsto (Q_{k+1} G(x_{0:k}, x_k + .))(u)$ is $\downarrow$ on $(-\infty, 0)$ and $\uparrow$ on $(0, +\infty)$.

&

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- (Second) backward induction $\Longrightarrow$ all functions Φ_k are convex.

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- (Second) backward induction $\Longrightarrow$ all functions Φ_k are convex.
- (Third) backward induction $\Longrightarrow$ $\Phi_k\leq\Psi_k$, $k=0:n-1$.

  First note that $\Phi_n=\Psi_n=F$. If $\Phi_{k+1}\leq\Psi_{k+1}$, then

$$\begin{aligned}\Phi_k(x_{0:k}) &= \big(Q_{k+1}\Phi_{k+1}(x_{0,k},x_k+.)\big)(\sigma_k(x_k))\\ &\leq \big(Q_{k+1}\Phi_{k+1}(x_{0:k},x_k+.)\big)(\theta_k(x_k))\\ &\leq \big(Q_{k+1}\Psi_{k+1}(x_{0:k},x_k+.)\big)(\theta_k(x_k))=\Psi_k(x_{0:k}).\end{aligned}$$

▷ Assume now that all functions σ_k are ≥ 0 and convex:

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\Phi_k(x_{0:k}) &= \big(Q_{k+1} \Phi_{k+1}(x_{0,k}, x_k + .)\big)(\sigma_k(x_k)) \\
&\leq \big(Q_{k+1} \Phi_{k+1}(x_{0:k}, x_k + .)\big)(\theta_k(x_k)) \\
&\leq \big(Q_{k+1} \Psi_{k+1}(x_{0:k}, x_k + .)\big)(\theta_k(x_k)) = \Psi_k(x_{0:k}).
\end{aligned}$$

End of discrete times

▷ If all $\theta_k \geq 0$ and convex:

This time, one shows that:

- the functions Ψ_k are convex, $k = 0, \ldots, n$
- $\Phi_n \leq \Psi_n \Longrightarrow \Phi_k \leq \Psi_k$, $k = 0, \ldots, n-1$.

Remark. The discrete time setting has its own interest.

Step 2 of the proof: Back to continuous time

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$\triangleright$ Euler scheme(s): Discrete time Euler scheme with step $\frac{T}{n}$, starting at x is an ARCH model. For $X^{(\sigma)}$: for $k = 0, \ldots, n-1$,

$$\bar{X}^{(\sigma),n}_{t^n_{k+1}} = \bar{X}^{(\sigma),n}_{t^n_k} + \sigma(t^n_k, \bar{X}^{(\sigma),n}_{t^n_k})(W_{t^n_{k+1}} - W_{t^n_k}),\ \bar{X}^{(\sigma),n}_0 = x$$

Set

$$Z_k = W_{t^n_k} - W_{t^n_{k-1}},\ k = 1, \ldots, n$$

$$\Downarrow$$

$$\boxed{\text{discrete time setting applies}}$$

Remark. Linear growth of σ and θ, implies

$$\forall\, p > 0, \qquad \sup_{n\ge 1} \Big\| \sup_{t\in[0,T]} |\bar{X}^{(\sigma),n}_t| \Big\|_p + \sup_{n\ge 1} \Big\| \sup_{t\in[0,T]} |\bar{X}^{(\theta),n}_t| \Big\|_p < +\infty.$$

From discrete to continuous time

▷ Interpolation ($n \geq 1$)

- *Piecewise affine interpolator* defined by

$$\forall\, x_{0:n} \in \mathbb{R}^{n+1},\ \forall\, k = 0, \ldots, n-1,\ \forall\, t \in [t_k^n, t_{k+1}^n], \quad .$$

$$i_n(x_{0:n})(t) = \frac{n}{T}\big((t_{k+1}^n - t)x_k + (t - t_k^n)x_{k+1}\big)$$

- $\widetilde{X}^{(\sigma),n} := i_n\Big((\bar{X}^{(\sigma),n}_{t_k^n})_{k=0:n}\Big)$ = piecewise affine Euler scheme.

▷ $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$ convex functional (with r-polynomial growth).

$$\forall\, n \geq 1, \quad F_n(x_{0:n}) := F\big(i_n(x_{0:n})\big), \quad x_{0:n} \in \mathbb{R}^{n+1}.$$

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- STEP 1 (DISCRETE TIME):

$$F \text{ convex} \Longrightarrow F_n \text{ convex}, \; n \geq 1.$$

Discrete time result implies: $\Big[\!\!\Big[\sigma(t_k^n,.) \leq \Big[\!\!\Big[\kappa(t_k^n,.)\Big]\!\!\Big] \leq \theta(t_k^n,.)\Big]\!\!\Big]$.

$$\mathbb{E}\, F\big(\widetilde{X}^{(\sigma),n}\big) = \mathbb{E}\, F_n\big((\bar{X}^{(\sigma),n}_{t_k^n})_{k=0:n}\big) \leq \mathbb{E}\, F_n\big((\bar{X}^{(\kappa),n}_{t_k^n})_{k=0:n}\big)$$
$$\leq \mathbb{E}\, F_n\big((\bar{X}^{(\theta),n}_{t_k^n})_{k=0:n}\big) = \mathbb{E}\, F\big(\widetilde{X}^{(\theta),n}\big).$$

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$$\leq \mathbb{E}\, F_n\big((\bar{X}^{(\theta),n}_{t_k^n})_{k=0:n}\big) = \mathbb{E}\, F\big(\widetilde{X}^{(\theta),n}\big).$$

- STEP 2 (TRANSFER): Key = Theorem 3.39, p.551, Jacod-Shiryaev's book (2^{nd} edition).

$$\widetilde{X}^{(\sigma),n} \xrightarrow{\mathcal{L}(\|.\|_{\sup})} X^{(\sigma)} \quad \text{as } n \to \infty.$$

$$\mathbb{E}\, F(X^{(\sigma)}) = \lim_n \mathbb{E}\, F\big(\widetilde{X}^{(\sigma),n}\big) \qquad (\textit{idem} \text{ for } X^{(\theta)}). \qquad \square$$

$\boxed{\text{The Euler scheme is a simulable approximation}}$

$\boxed{\text{which preserves convex order.}}$

Application I : Local Volatility models (functional peacocks).

- New notations (σ, θ are now true volatility)

$$dS_t = S_t\,\widetilde{\sigma}(t,S_t)dW_t,\; S_0 = s_0 > 0, \qquad (r=0)$$

where $\widetilde{\sigma} : [0,T]\times\mathbb{R}\to\mathbb{R}$ is a bounded continuous function.

- The weak solution satisfies

$$S_t^{(\widetilde{\sigma})} = s_0 e^{\int_0^t \widetilde{\sigma}(s,S_s^{(\sigma)})dB_s - \frac{1}{2}\int_0^t \sigma^2(s,S_s^{(\sigma)})ds} > 0.$$

- Idem for θ.
- We assume that $0\le\widetilde{\sigma}\le\widetilde{\kappa}\le\widetilde{\theta}$ and $\kappa : x\mapsto x\widetilde{\kappa}(t,x)$ is convex on the whole real line.

Example of application: El Karoui-Jeanblanc-Shreve's Theorem.

Theorem (Extension of El Karoui et al. theorem., P. 2016)

If there exists a function $\widetilde{\kappa} : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\kappa : x \mapsto x\widetilde{\kappa}(t, x)$ *extends into an* $\mathbb{R}_+$*-valued convex function on* $\mathbb{R}$ *satisfying*

(a) *Partitioning:* $0 \le \widetilde{\sigma}(t, .) \le \widetilde{\kappa}(t, .) \le \widetilde{\theta}(t, .)$ *on* $\mathbb{R}_+$, $t \in [0, T]$,

or

(b) *Dominating:* $|\widetilde{\sigma}(t, .)| \le \widetilde{\theta}(t, .) = \widetilde{\kappa}(t, .)\ \ t \in [0, T]$.

If $f : \mathbb{R} \to \mathbb{R}$ *convex function and* μ *is a (finite) signed measure on* $[0, T]$

$$\mathbb{E}\, f\left(\int_0^T S_s^{(\widetilde{\sigma})}\mu(ds)\right) \le \mathbb{E}\, f\left(\int_0^T S_s^{(\widetilde{\theta})}\mu(ds)\right) \in (-\infty, +\infty].$$

and more generally, for every every functional F *with* r*-polynomial growth*

$$\mathbb{E}\, F\left(S^{(\widetilde{\sigma})}\right) \le \mathbb{E}\, F\left(S^{(\widetilde{\theta})}\right) \in \mathbb{R}.$$

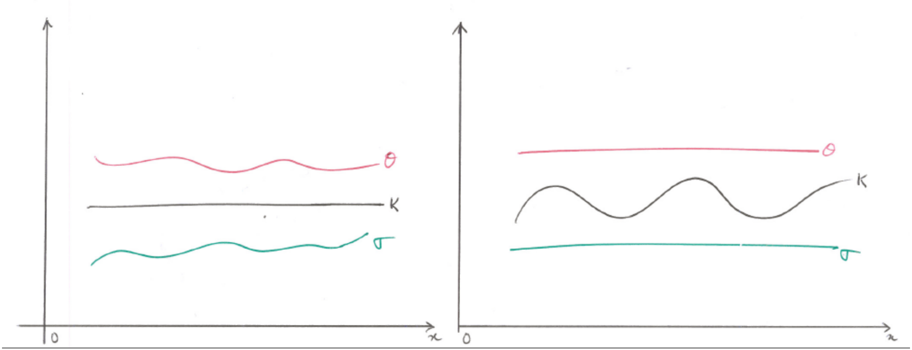


Figure: Left: flat partitioning. Right: flat bounding (El Karoui et al.).

- The method of proof applies to American style options, Lévy driven diffusions, stochastic integrals, etc.
- 1D-Misltein scheme.

Application II: Concave Local Vol. models (with A. Fadili)

- Concave Local Volatility (CLV) models ($\ni$ CEV):

  Set $\sigma(x) = x\,\widetilde{\sigma}(x) > 0$ on $(0, +\infty)$, $\sigma(x) = 0$, $x \leq 0$.

$$dS_t = \sigma(S_t)dW_t,\ S_0 = s_0 > 0, \quad \sigma \text{ concave and } > 0$$

  such that the (unique weak) solution satisfies $S_t \geq 0$, $t \in [0, T]$.

- Then, for every fixed $u > 0$, the concavity property implies

$$\sigma(x) \leq \Big(\sigma(u) + \sigma'(u)(x-u)\Big)_+,\ x \in \mathbb{R}$$

  so that, if we set

$$dX_t^{(u)} = \Big(\sigma(u) + \sigma'(u)(X_t^{(u)} - u)\Big)_+ dW_t,\ X_0^{(u)} = s_0$$

  then, for every convex vanilla payoff $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$

$$\boxed{\mathbb{E}\,\varphi(S_T) \leq \inf_{u>0} \mathbb{E}\,\varphi\big(X_T^{(u)}\big).}$$

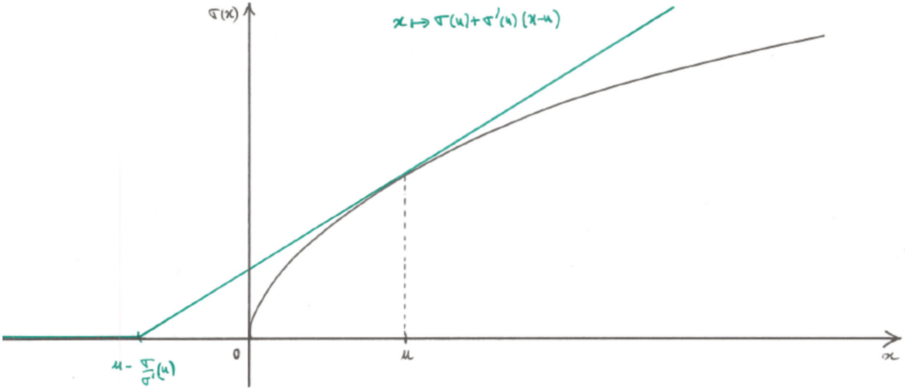


Figure: Black-Scholes convex domination of a Local Vol. model

- Set $\theta(u) := \frac{\sigma'(u)}{\sigma(u)} - u > 0$ (by concavity). Hence

$$X_t^{(u)} + \theta(u) = \underbrace{s_0 + \theta(u)}_{>0} + \int_0^t \sigma'(u)\big(X_s^{(u)} + \theta(u)\big)_+ dW_s$$

i.e., $Y_t^{(u)} = X_t^{(u)} + \theta(u)$, satisfies the Black-Scholes dynamics

$$Y_t^{(u)} = Y_0^{(u)} + \sigma'(u) \int_0^t Y_s^{(u)} dW_s.$$

- Example: if $\varphi(x) = (x-K)_+$ is a vanilla Call payoff

$$\mathbb{E}\,(S_T - K)_+ \le \inf_{u>0} \mathrm{Call}_{BS}\Big(s_0 + \theta(u), K + \theta(u), \sigma'(u)\Big)$$

Proposition (Tractable upper-bound)

One has

(*i*) $u \mapsto \mathbb{E}\,(Y_T^{(u)} - K)_+$ *is differentiable and* $\frac{\partial}{\partial u}\mathbb{E}\,(Y_T^{(u)} - K)_+ \geq 0$ *on* $\big[\max(s_0, K), +\infty\big)$

(*ii*) *Hence*

$$\mathbb{E}\,(S_T - K)_+ \leq \min_{0 \leq u \leq \max(s_0, K)} \mathrm{Call}_{BS}\big(s_0 + \theta(u), K + \theta(u), \sigma'(u)\big)$$

leading to a faster search for the argmin.

Practitioner's corner: – In fact $u_{\min}$ lies not far from s_0 and K.

– Exploration starting from $\frac{s_0+K}{2}$.

When the drift comes back into the game: non-decreasing convex order for diffusions

Theorem (Extended Hajek's Theorem, P. 2016, *Sém. Prob. XLVIII,*)

Let $\sigma, \theta \in \mathcal{C}_{lin_x}([0,T]\times\mathbb{R},\mathbb{R})$. *Let* $X^{(\sigma)}$ *and* $X^{(\theta)}$ *be the* unique *weak solutions to*

$$\begin{aligned} dX_t^{(\sigma)} &= b(t,X_t^{(\sigma)})dt + \sigma(t,X_t^{(\sigma)})dW_t^{(\sigma)},\ X_0^{(\sigma)} = x \\ dX_t^{(\theta)} &= b(t,X_t^{(\theta)})dt + \theta(t,X_t^{(\theta)})dW_t^{(\theta)},\ X_0^{(\theta)} = x, \end{aligned}$$

with $(W_t^{(\cdot)})_{t\in[0,T]}$ *is a standard dim B.M. Then,*

(i) $b(t,.)$ *convex* $t\in[0,T]$ *and* $|b(t,x)|\le C(1+|x|)$.

(ii) κ*-partitioning or dominating assumption.*

▷ *Then, for every* $f:\mathbb{R}\to\mathbb{R}$, *convex and non-decreasing, with polynomial growth,*

$$\mathbb{E}\, f(X_T^{(\sigma)}) \le \mathbb{E}\, f(X_T^{(\theta)}).$$

Multidimensional extension (with A. Fadili, 2017)

- Pre-order $\preceq$ on $\mathcal{M}(d,q,\mathbb{R})$: let A, $B\in\mathcal{M}(d,q,\mathbb{R})$.

$$A\preceq B \quad \text{if} \quad AA^*\leq BB^* \text{ in } \mathcal{S}(d,\mathbb{R}).$$

- $\preceq$-Convexity: A function $\phi:\mathbb{R}^d\to\mathcal{M}(d,q,\mathbb{R})$ is $\preceq$-convex if: for every x, $y\in\mathbb{R}^d$, and every $\lambda\in[0,1]$,

$$\phi(\lambda x+(1-\lambda)y))\preceq\lambda\phi(x)+(1-\lambda)\phi(y).$$

Proposition

Let A, $B\in\mathcal{M}(d,q,\mathbb{R})$ such that $A\preceq B$. Let $Z\sim\mathcal{N}(0,I_q)$. Then, for every $\preceq$-convex function $f:\mathbb{R}^d\to\mathbb{R}$,

$$\mathbb{E}\,f(AZ)\leq\mathbb{E}\,f(BZ).$$

- The former theorem formally extended remains valid with these definitions for diffusion models of the form

$$dX_t=\sigma(X_t)dW_t,\ \sigma:\mathbb{R}^d\to\mathcal{M}(d,q,\mathbb{R}),\ Z\sim\mathcal{N}(0,I_q).$$

Theorem (with A. Fadili, 2017)

Let $\sigma, \theta \in \mathcal{C}_{lin_x}\big([0,T]\times\mathbb{R}, \mathcal{M}(d,q,\mathbb{R})\big)$, $W^{(\sigma)}$, $W^{(\theta)}$ *q-S.B.M.. Let* $X^{(\sigma)}$ *and* $X^{(\theta)}$ *be the* unique *weak solutions to*

$$\begin{aligned} dX_t^{(\sigma)} &= \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},\ X_0^{(\sigma)} = x \\ dX_t^{(\theta)} &= \theta(t, X_t^{(\theta)})dW_t^{(\theta)},\ X_0^{(\theta)} = x, \quad (W_t^{(\cdot)})_{t\in[0,T]} \text{ standard B.M.} \end{aligned}$$

(a) $\triangleright$ *If there exists a partitioning function* $\kappa \in \mathcal{C}_{lin_x}([0,T]\times\mathbb{R}, \mathcal{M}(d,q,\mathbb{R}))$ *s.t.*

$$\begin{cases} (i) & \kappa(t,.) : \mathbb{R}^d \to \mathcal{M}(d,q,\mathbb{R}) \text{ is } \preceq\text{-convex for every } t\in[0,T], \\ (ii) & \sigma \preceq \kappa \leq \theta. \end{cases}$$

$\triangleright$ *Then, for every* $F : \mathcal{C}([0,T],\mathbb{R}^d) \to \mathbb{R}$, *convex, with* $\|\,.\,\|_{\sup}$*-polynomial growth (hence* $\|\,.\,\|_{\sup}$*-continuous). Then*

$$\mathbb{E}\, F(X^{(\sigma)}) \leq \mathbb{E}\, F(X^{(\theta)}).$$

(b) *Domination: If* $\sigma \preceq \theta = \kappa$ *(hence convex), the conclusion still holds true.*

Extensions

This provides as systematic approach whcih sucessfully works with

- Jump diffusion models,
- Path-dependent American style options,
- BSDE (without "Z" in the driver),
- ...

Another kind of application: MOT

- Let $X_{0:n}$ be a martingale on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution $\mu \in \mathcal{P}((\mathbb{R}^d)^{n+1})$ and marginal distributions μ_k, $k = 0 : n$.
- Let $c : (\mathbb{R}^d)^{n+1} \to \mathbb{R}_+$ be a cost/payoff fucntion (exotic option...).
- Assume the marginal distributions are fixed (as the result of a calibrations on vanilla options). Solve

$$(MOT) \equiv \inf / \sup \Big\{ \mathbb{E}c(X_{0:n}),\, X \text{ martingale},\, X_k \sim \mu_k \Big\}$$

- Yields bounds on the exotic option premium.

- Think of $X = (X_{0:n})$ as an Euler time discretization scheme of a diffusion.
- Then

$$X_{k+1} = X_k + \vartheta_k(X_k)Z_{k+1},\ k = 0 : n-1,\ Z_k \text{ i.i.d.}$$

with $\vartheta_k : \mathbb{R}^d \to \mathbb{M}_{d,q}$, $Z_1 \sim \mathcal{N}(0, I_q)$.
- The (MOT) problem cannot be solved as set : it requires
  - a space discretization:

$$(\widehat{X}_0, \widehat{X}_1, \cdots, \widehat{X}_n) \simeq (X_0, X_1, \cdots, X_n)$$

    where each $\widehat{X}_k$ takes finitely many values.
  - satisfying monotony for convex order (to avoid arbitrages):

$$\widehat{X}_0 \leq_{cvx} \widehat{X}_1 \leq_{cvx} \cdots \leq_{cvx} \widehat{X}_n,$$

  - comparison results for convex order with respect to X,
  - A complexity kept under control.

A solution: Dual quantization at level $N \geq 1$!

- Let $Y : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$, $Y \sim \mu$, be a random vector lying in $L^\infty(\mathbb{P})$.
- Let N denote a fixed level $N \geq 1$.
- The optimal dual quantization problem introduced by [Wilbertz-P., '12] reads

$$d_{p,N}(Y) = \inf_{\widehat{X}} \Big\{ \big\| Y - \widehat{Y} \big\|_p : \widehat{X} : (\Omega \times \Omega_0, \mathcal{A} \otimes \mathcal{A}_0, \mathbb{P} \otimes \mathbb{P}_0) \to \mathbb{R}^d, \\ \mathrm{card}\, \widehat{Y}(\Omega \times \Omega_0) \leq N \text{ and } \mathbb{E}(\widehat{Y} | Y) = Y \Big\}$$

or, equivalently,

$$d_{p,N}(\mu) = \inf_{X} \Big\{ \| Y - V \|_p,\ (Y, V) : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d \times \mathbb{R}^d, \\ Y \sim \mu,\ \mathbb{E}(Y \,|\, V) = Y,\ \mathrm{card}\big(V(\Omega)\big) \leq N \Big\}.$$

Theorem (P.-Wilbertz, *SINUM '12*)

Assume $Y \in L^\infty(\Omega, \mathcal{A}, \mathbb{P})$ *with* $\mathrm{card}\big(Y(\Omega)\big) \geq N\}$.

(*a*) *Let* $\Gamma \subset \mathbb{R}^d$, $\Gamma \supset \mathrm{conv}\big(\mathrm{supp}(\mathcal{L}(Y)\big)$ *with points in general position. Then*

$$\widehat{Y}^{del,\Gamma} = \mathrm{Proj}^{del}_{\Gamma}(Y, U_{[0,1]})$$

$$\textit{with} \qquad \mathrm{Proj}^{del}_{\Gamma}(\xi, u) = \sum_{k=1}^{m}\Bigg[\sum_{i=1}^{N} x_i \cdot \mathbf{1}_{\Big\{\sum_{j=1}^{i-1}\lambda^k_j(\xi) \leq u < \sum_{j=1}^{i}\lambda^k_j(\xi)\Big\}}\Bigg]\mathbf{1}_{D_k(\Gamma)}(\xi).$$

where $(D_k(\Gamma))_{1\leq k\leq m}$ *is a Delaunay hyper-triangulation of* $\mathrm{conv}(\Gamma^{*,N})$ *satisfies the dual stationarity equation*

$$\mathbb{E}\big(\widehat{Y}^{del,\Gamma}\,|\,Y\big) = Y$$

and $\big\|Y - \widehat{Y}^{del,\Gamma}\big\|_2 = \inf_X \Big\{\|Y - V\|_p,\, (Y,V) : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d \times \Gamma,$

$$Y \sim \mu,\ \mathbb{E}(Y\,|\,V) = Y,\ \mathrm{card}\big(V(\Omega)\big) \leq N\Big\}.$$

Theorem

(b) *The above infimum in* $d_{p,N}(X)$ *is always a minimum: there exists a grid* $\Gamma^{*,N} = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ *of size* N *whose points are in general position such that*

$$d_{p,N}(Y) = \|Y - \widehat{Y}^{del,N}\|_p \quad \textit{with} \quad \widehat{Y}^{del,N} = \mathrm{Proj}^{del}_{\Gamma^{*,N}}(Y, U_{[0,1]})$$

The Delaunay projection/splitting operator

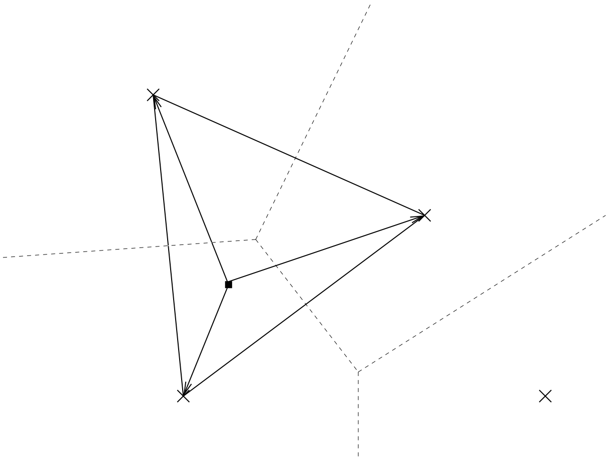
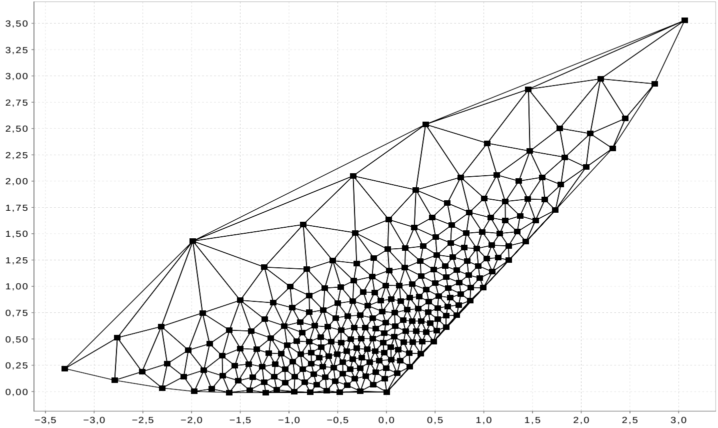
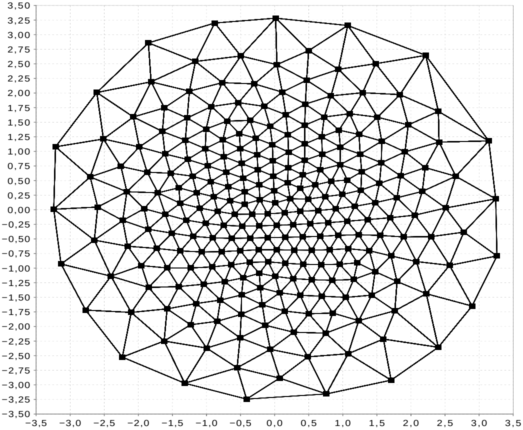


Figure: Delaunay splitting of $X(\omega)$.

Dual quantization of $(B_1, \sup_{t\in[0,1]} B_t)$ (truncated)



Dual quantization of $(B_1, \sup_{t\in[0,1]} B_t)$ (truncated)



Doubly quantized scheme (martingale ARCH model) I

- The (martingale) dynamics. Set

$$X_k = F_{k-1}(X_{k-1}, Z_k),\ (Z_k)\ i.i.d; \sim \mathcal{N}(0; I_q)$$

with $F_{k-1}(x, z) \mapsto x + \vartheta_{k-1}(x)z\ k = 1, \ldots, n-1.$

- Pre-processing. Discretize the Gaussian white noise by $\varphi : \mathbb{R}^d \to \Gamma^Z \subset \mathbb{R}^q$,

$$\breve{Z}_k = \varphi(Z_k)\ \text{ s.t. }\ \mathbb{E}\,\breve{Z}_k = 0,\ Z_k - \breve{Z}_k \perp_{L^2} \breve{Z}_k,$$

Example: Optimal Voronoi quantization i.e. $\breve{Z}_k = \mathrm{Proj}^{vor}_{\Gamma^Z}(Z_k)$, $k = 1 : n$ which satisfies

$$\mathbb{E}(Z_k \,|\, \breve{Z}_k) = \breve{Z}_k,\ k = 1, \ldots, n.$$

Doubly quantized scheme (martingale ARCH model) II

- The doubly quantized scheme. Define by induction, based on grids $\Gamma_0, \ldots, \Gamma_n$,
  (i) $\widehat{X}_0 = \mathrm{Proj}^{vor}_{\Gamma_0}(X_0)$ (Voronoi "nearest neighbour" projection)

Doubly quantized scheme (martingale ARCH model) II

- The doubly quantized scheme. Define by induction, based on grids $\Gamma_0, \dots, \Gamma_n$,
  (i) $\widehat{X}_0 = \mathrm{Proj}^{vor}_{\Gamma_0}(X_0)$ (Voronoi "nearest neighbour" projection)
  (ii) $\widetilde{X}_k = F_{k-1}(\widehat{X}_{k-1}, \check{Z}_k), \quad k = 1:n$

Doubly quantized scheme (martingale ARCH model) II

- The doubly quantized scheme. Define by induction, based on grids $\Gamma_0, \ldots, \Gamma_n$,
  (i) $\widehat{X}_0 = \mathrm{Proj}^{vor}_{\Gamma_0}(X_0)$ (Voronoi "nearest neighbour" projection)
  (ii) $\widetilde{X}_k = F_{k-1}(\widehat{X}_{k-1}, \check{Z}_k), \quad k = 1:n$
  (iii) $\widehat{X}_k = \mathrm{Proj}^{del}_{\Gamma_k}(\widetilde{X}_k, U_k)$

  where

$$(U_{1:n}) \text{ is i.i.d. } U([0,1])\text{-distributed}, \perp\!\!\!\perp (Z_{1:n}), \perp\!\!\!\perp X_0.$$

Proposition (Jourdain-P., '19)

(a) *The sequence $(\widehat{X}_k)_{k=0:n}$ is martingale Markov chain so that*

$$\widehat{X}_0 \leq_{cvx} \widehat{X}_1 \leq_{cvx} \cdots \leq_{cvx} \widehat{X}_n.$$

(b) *Moreover, if the θ_k are convex (in a matrix sense of d or $q \geq 2$) then*

$$\forall\, k = 0:n, \quad \widehat{X}_k \leq_{cvx} X_k$$

A short proof of (a)

- $\widehat{X}_0 = \mathrm{Proj}^{vor}_{\Gamma_0}(X_0)$ is Γ_0-valued hence has compact support.
- If $\widehat{X}_{k-1}$ have compact support, then $\widetilde{X}_k$ has compact support since $\check{Z}_k$ has. Hence there exists $\Gamma_n \supset \mathrm{conv}\big(\widetilde{X}_k(\Omega)\big)$ and one can define $\widehat{X}_k = Proj^{del}_{\Gamma_k}(\widetilde{X}_k, U_k)$.
- The Markov property is obvious since $\widehat{X}_k = \widehat{F}_{k-1}\big(\widehat{X}_{k-1}, (U_k, Z_k)\big)$.
- Finally, by the universal dual stationarity property

$$\mathbb{E}\big(\widehat{X}_k \,|\, \sigma(X_0, Z_\ell, \ell = 1 : k-1, U_\ell, \ell = 1, k-1)\big) = \big[\mathbb{E}\,\mathrm{Proj}^{del}_{\Gamma_k}(x, U_k)\big]_{|x=\widetilde{X}_k}$$

and the martingality of the "kernel" F_{k-1} yields

$$\mathbb{E}\big(\widetilde{X}_k \,|\, \mathcal{F}^{X_0,Z,U}_{k-1}\big) = \widehat{X}_{k-1}$$

so that

$$\mathbb{E}\big(\widehat{X}_k \,|\, \mathcal{F}^{X_0,Z,U}_{k-1}\big) = \widehat{X}_{k-1}$$

Error bound

Theorem (Jourdain-P. '19)

Assume all ϑ_k are $[\vartheta]_{\rm Lip}$-Lipschitz continuous and all quantizations are optimal/optimized at respective levels $|\Gamma_k| = N_k$ and $|\Gamma^Z| = N$. For every $k = 0 : n$,

$$
\|\widehat{X}_k - X_k\|_2^2 \le (C^{vor}_{d,\eta})^2 \big(1 + q[\vartheta]^2_{\rm Lip}\big)^k \frac{\sigma^2_{2+\eta}(X_0)}{N_0^{2/d}}
$$

$$
+ \sum_{\ell=1}^{k} \big(1 + q[\vartheta]^2_{\rm Lip}\big)^{k-\ell} \left[\|\vartheta_{\ell-1}(X_{\ell-1})\|_2^2 (C^{vor}_{q,\eta})^2 \frac{\sigma^2_{2+\eta}(Z)}{N^{2/q}} + (\widetilde{C}^{del}_{d,\eta})^2 \frac{\sigma^2_{2+\eta}(\widetilde{X}_k)}{N_k^{2/d}} \right].
$$