Continuous time Principal Agent and optimal planning

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(Static) Principal-Agent Problem

- Principal delegates management of output process X,
  only observes X

- Agent devotes effort $a \Longrightarrow X^a$, chooses optimal effort by

$$V_A := \max_a \mathbb{E}\, U_A\big(\qquad - c(a)\big)$$

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$$V_A(\xi) := \max_a \mathbb{E}\, U_A\big(\xi(X^a) - c(a)\big) \quad \Longrightarrow \quad \hat{a}(\xi)$$

- Principal chooses optimal contract by solving

$$\max_\xi \mathbb{E}\, U_P\big(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})\big) \quad \text{under constraint} \quad V_A(\xi) \geq \rho$$

$\Longrightarrow$ Non-zero sum Stackelberg game

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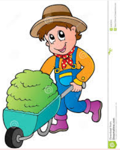
Principal-Agent problem formulation



Contract $\mathbf{C} = (\tau, \pi, \xi)$

τ $\mathbb{F}-$stopping time, π $\mathbb{F}-$adapted, and ξ $\mathcal{F}_\tau-$mble

Agent problem

$$V_0^A(\mathbf{C}) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[\xi(X) + \int_0^\tau \big(\pi_t - c_t(\nu_t)\big)dt\Big]$$

$\mathbb{P} \in \mathcal{P}$: weak solution of Output process :

$dX_t = b_t(X, \nu_t)dt + \sigma_t(X, \nu_t)dW_t^{\mathbb{P}}$ $\mathbb{P}-$a.s.

for some ν valued in U

Principal problem choose among acceptable contracts

$$\Xi_\rho := \{\mathbf{C} : V_0^A(\mathbf{C}) \geq \rho\}$$

best contract, given Agent's optimal response $\mathbb{P}^*(\mathbf{C})$

$$V_0^P := \sup_{\mathbf{C}\in\Xi_\rho} \mathbb{E}^{\mathbb{P}^*(\mathbf{C})}\Big[U\Big(\ell(X) - \xi(X) - \int_0^\tau \pi_t(X)dt\Big)\Big]$$

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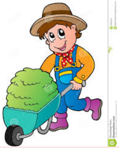

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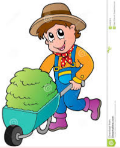

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A general solution approach

- Path-dependent Hamiltonian for the Agent problem :

$$H_t^\pi(\omega,z,\gamma) := \sup_{u\in U}\Big\{b_t(\omega,u)\cdot z+\frac{1}{2}\sigma_t\sigma_t^\top(\omega,u):\gamma+\pi_t(\omega)-c_t(\omega,u)\Big\}$$

- For $Y_0\in\mathbb{R}$, Z,Γ $\mathbb{F}^X-$ prog meas, define $\mathbb{P}-$a.s. for all $\mathbb{P}\in\mathcal{P}$

$$Y_t^{Z,\Gamma} = Y_0+\int_0^t Z_s\cdot dX_s+\frac{1}{2}\Gamma_s : d\langle X\rangle_s - H_s^\pi(X,Z_s,\Gamma_s)ds$$

Proposition

$V_A(\tau,\pi,Y_\tau^{Z,\Gamma}) = Y_0$. Moreover $\mathbb{P}^*$ is optimal iff

$$\nu_t^* = \operatorname*{Argmax}_{u\in U} H_t^\pi(Z_t,\Gamma_t) = \hat{\nu}(Z_t,\Gamma_t)$$

Proof classical verification argument in stochastic control

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Principal problem restricted to revealing contracts

Dynamics of the pair (X, Y) under "optimal response"

$$dX_t = b_t\big(X, \hat{\nu}(Z_t, \Gamma_t)\big)dt + \sigma_t\big(X, \hat{\nu}(Z_t, \Gamma_t)\big)dW_t$$

$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \tfrac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t^{\pi}(X, Z_t, \Gamma_t)dt$$

is a (1 state augmented) controlled SDE with controls (π, Z, Γ)

$\Longrightarrow$ Principal's value function under revealing contracts :

$$V_P \geq V_0(X_0, Y_0) := \sup_{\substack{(\tau,\pi) \\ (Z,\Gamma)\in\mathcal{V}}} \mathbb{E}\Big[U\Big(\ell(X) - Y_\tau^{Z,\Gamma} - \int_0^\tau \pi_t dt\Big)\Big], \ \forall\ Y_0 \geq \rho$$

where $\mathcal{V} := \Big\{(Z,\Gamma) : \ Z \in \mathbb{H}^2(\mathcal{P}) \text{ and } \ \mathcal{P}^*\big(Y_T^{Z,\Gamma}\big) \neq \emptyset\Big\}$

Reduction to standard control problem

Theorem

Assume $\mathcal{V} \neq \emptyset$*. Then*

$$V_0^P = \sup_{Y_0 \geq \rho} V_0(X_0, Y_0)$$

Given maximizer Y_0^**, the corresponding optimal controls* $(\tau^*, \pi^*, Z^*, \Gamma^*)$ *induce an optimal contract* $\mathbf{C}^* = (\tau^*, \pi^*, \xi^*)$ *with*

$$\xi^* = Y_0^* + \int_0^T Z_t^* \cdot dX_t + \frac{1}{2}\Gamma_t^* : d\langle X\rangle_t - H_t^{\pi^*}(X, Z_t^*, \Gamma_t^*)dt$$

Sannikov '08
Cvitanić, Possamaï & NT '15
Lin, Ren, NT & Yang '19

Comments on the theorem

Examples of volatility control problems

- Portfolio optimization

$$dV_t = \theta_t \cdot dS_t$$

- Demand-Response programs in electricity retail market

$$dX_t = \alpha_t dt + \beta_t \cdot dW_t$$

Open to many extensions

- agent may also choose optimally to quit (Sannikov)
- many agents, many principals under competition (Possamaï &...)
- Limited liability (Possamaï & Villeneuve)

Recall the subclass of contracts

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$\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$

To prove the main result, it suffices to prove the **representation**

for all $\xi \in ??$ $\exists\,(Y_0, Z, \Gamma)$ s.t. $\xi = Y_T^{Z,\Gamma}$, $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$

OR, weaker sufficient condition :

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The uncontrolled volatility case

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($\mathcal{P}$ dominated set of measures by Girsanov)

Representation problem reduces to

$$Y_T^{Z,0} = \xi, \quad \mathbb{P}_0 - \text{a.s.} \quad \text{Backward SDE...} \qquad \square$$

Pardoux & Peng '90, El Karoui, Peng & Quenez '96

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- $H_t(\omega, y, z, \gamma)$ non-decreasing and convex in γ, the

$$H_t(\omega, y, z, \gamma) \quad = \quad \sup_{\sigma} \left\{ \frac{1}{2}\sigma^2 : \gamma - H_t^*(\omega, y, z, \sigma) \right\}$$

- Let $\hat{\sigma}_t^2 := \frac{d\langle X \rangle}{dt}$, and introduce

$$k_t := H_t(Y_t, Z_t, \Gamma_t) - \frac{1}{2}\hat{\sigma}_t^2 : \Gamma_t + H_t^*(Y_t, Z_t, \hat{\sigma}_t) : \quad \geq 0 \quad \text{and } \text{“}\inf_{\mathbb{P} \in \mathcal{P}} k_t = 0\text{''}$$

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Wellposedness of random horizon 2nd order backward SDE

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} F_s(Y_s, Z_s, \hat{\sigma}_s)ds - \int_{t\wedge\tau}^{\tau} Z_s \cdot dX_s + \int_{t\wedge\tau}^{\tau} dK_s, \quad \mathcal{P} - \text{q.s.}$$

K non-decreasing, and $\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\int_{s\wedge\tau}^{t\wedge\tau} dK_r\right] = 0$, $s \le t$

Theorem (Y. Lin , Z. Ren, NT & J. Yang '18)

Assume $\exists\ \rho > -\mu,\ \ q > 1 : \mathcal{E}^L\left[\left|e^{\rho\tau}\xi\right|^q\right] + \mathcal{E}^L\left[\left(\int_0^\tau \left|e^{\rho t} f_t^0\right|^2 ds\right)^{\frac{q}{2}}\right] < \infty$

Then, Random horizon 2BSDE has a unique solution (Y, Z) *with*

$$Y \in \mathcal{D}^p_{\eta,\tau}, \quad Z \in \mathcal{H}^p_{\eta,\tau} \quad \text{for all} \quad \eta \in [-\mu, \rho),\ \ p \in [1, q)$$

$$\|Y\|^p_{\mathcal{D}^p_{\eta,\tau}} := \mathcal{E}^L\left[\sup_{t\le\tau}\left|e^{\eta t} Y_t\right|^p\right], \quad \|Z\|^p_{\mathcal{H}^p_{\eta,\tau}} := \mathcal{E}^L\left[\left(\int_0^\tau \left|e^{\eta t}\,\widehat{\sigma}_t^{\mathrm{T}} Z_t\right|^2 dt\right)^{\frac{p}{2}}\right]$$

Extends Soner, NT & Zhang '12 and Possamaï, Tan & Zhou '18
Closely connected to G-BSDE, Hu, Ji, Peng & Song '14

Mean field games

Consider a croud of agents in **MFG equilibrium** :

$$V_0^A(\mu,\xi) := \sup_{\mathbb{P}\in\mathcal{P}} J(\mu,\xi,\mathbb{P}) = J\big(\mu,\xi,\hat{\mathbb{P}}_\xi^\mu\big)$$

where $$J(\mathbb{P},\mu,\xi) := \mathbb{E}^{\mathbb{P}}\Big[\xi(X) - \int_0^T c_t(\mu_t,\alpha_t,\beta_t)dt\Big]$$

and $\mathbb{P}\in\mathcal{P}$ is weak solution of controlled process :

$$\mathbb{P}\circ(X_0)^{-1} = \mu_0, \text{ and } dX_t = \sigma_t(X,\beta_t)\big[\lambda_t(X,\alpha_t)dt + dW_t^{\mathbb{P}}\big], \ \mathbb{P}-\text{a.s.}$$

Definition (Mean field game equilibrium)

$\hat{\mu}$ is an MFG equilibrium if $\hat{\mathbb{P}}_\xi^{\hat{\mu}}\circ(X_t)^{-1} = \hat{\mu}_t$, for all $t\le T$

P.L. Lions' Planning Problem

- uncontrolled diffusion $\sigma_t(\beta) = I_d$
- Markov setting : $\lambda_t(\omega) = \lambda_t(\omega_t)$, c..., and $\xi(\omega) = g(\omega_T)$

Planning Problem

• Let μ_0, ν be given probability measures on $\mathbb{R}^d$
• Find $g : \mathbb{R}^d \longmapsto \mathbb{R}$ so that

MFG equilibrium $\hat{\mu}$ exists and $\hat{\mu}_0 = \mu_0$, $\hat{\mu}_T = \nu$

Interpretation : optimal transport, regulation

Start from croud distributed as μ_0. Choose an appropriate incentive cost $g : \mathbb{R}^d \longmapsto \mathbb{R}$ so as to force the MFG equilibrium to the target distribution ν at time T.

$\Longrightarrow$ Unique solution exists for any pair (μ, ν)...
Lions '10, Achdou Y, Camilli F & Capuzzo Dolcetta '12, Porretta '14

Path-dependent formulation of the Planning Problem

Allow the incentive cost ξ to be path-dependent

Path-dependent Planning Problem

- Let μ_0, ν be given probability measures on $\mathbb{R}^d$
- Find $\xi : \Omega \longmapsto \mathbb{R}$, $\mathcal{F}_T-$measurable, so that

$$\text{MFG equilibrium } \hat{\mu} \text{ exists and } \hat{\mu}_0 = \mu_0, \quad \hat{\mu}_T = \nu$$

- More freedom for the choice of incentive regulation
- Multiple solutions, in general

MFG equilibria with varying path-dependent cost ξ

Forward description of MFG equilibria

For all controls (Z, Γ), let $\xi^{Z,\Gamma} := Y_T^{Z,\Gamma}$, where $Y^{Z,\Gamma}$ is defined by the McKean-Vlasov controlled process

$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t\big(Y_t^{Z,\Gamma}, Z_t, \Gamma_t, \mu_t\big)$$

$\mu_t = \mathbb{P} \circ X_t^{-1}$ distribution of X_t with controls defined as maximizers of H

$$dX_t = \nabla_z H_t(\cdots)dt + \big[2\nabla_\gamma H_t(\cdots)\big]^{1/2} dW_t$$

- Multiple solutions, in general
- Skorohod embedding problem is a particular case :
  - planning exists iff $\mu_0 \preceq \nu$ in convex order
  - Many solutions exist... sometimes corresponding to various optimization criteria !

Optimal Planning Problem

MFG transport plans from μ_0 to ν in $\mathrm{Prob}(\mathbb{R}^d)$

- $\Xi(\mu_0, \nu)$: $\xi \in \Lambda^0(\Omega, \mathcal{F}_T)$ s.t. there exists an MFG equilibrium μ satisfying $\hat{\mathbb{P}}^{\mu}_{\xi} \circ (X_T)^{-1} = \nu$
- Given the planner criterion $\phi : \Omega \times \Lambda^0(\Omega, \mathcal{F}_T) \longrightarrow \mathbb{R}$, solve

$$V^{\mathrm{P}} \quad := \quad \sup_{\xi \in \Xi(\mu_0, \nu)} \mathbb{E}^{\hat{\mathbb{P}}^{\mu}_{\xi}} \Big[\phi(X, \xi(X)) \Big]$$

Theorem (Z. Ren, X. Tan & NT)

Planner problem can be restricted to forward MV transport plans

$$V^{\mathrm{P}} = \sup_{Z, \Gamma:\ \hat{\mathbb{P}}^{Z,\Gamma} \circ (X_T)^{-1} = \nu} \mathbb{E}^{\hat{\mathbb{P}}^{Z,\Gamma}} \left[\phi(X, Y_T^{Z,\Gamma}) \right]$$

BON ANNIVERSAIRE NICOLE

Nonlinear expectation operators

$\mathcal{P}^0$: subset of local martingale measures, i.e.

$$dX_t = \sigma_t dW_t, \quad \mathbb{P} - \text{a.s. for all} \quad \mathbb{P} \in \mathcal{P}^0$$

$\Longrightarrow$ Nonlinear expectation

$$\boxed{\mathcal{E} := \sup_{\mathbb{P}\in\mathcal{P}^0} \mathbb{E}^{\mathbb{P}}}$$

Similarly, $\mathcal{P}^L$: subset of measures $\mathbb{Q}^\lambda$ such that

$$dX_t = \sigma_t(\lambda_t dt + dW_t), \quad \mathbb{Q} - \text{a.s. for some} \quad \lambda, \ \mathbb{F} - \text{adapted}, \ |\lambda| \le L$$

$\Longrightarrow$ Another nonlinear expectation

$$\boxed{\mathcal{E}^L := \sup_{\mathbb{P}\in\mathcal{P}^L} \mathbb{E}^{\mathbb{Q}}}$$

$\mathcal{E}$ and $\mathcal{E}^L$ will play the role of Sobolev norms...

Nonlinearity

Assumptions on $F : \mathbb{R}_+ \times \omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_+^d \longrightarrow \mathbb{R}$

(C1$_L$) Lipschitz in $(y, \sigma z)$:

$$\big|F(., y, z, \sigma) - F(., y', z', \sigma)\big| \le L\,\big(|y - y'| + \big|\sigma(z - z')\big|\big)$$

(C2$_\mu$) Monotone in y :

$$(y - y') \cdot \big[F(., y, .) - F(., y', .)\big] \le -\mu\, |y - y'|^2$$

Denote $f_t^0 := F_t\big(0, 0, \widehat{\sigma}_t\big)$

Remark Deterministic finite horizon $\tau = T$: $(\mathrm{C2})_\mu$ not needed
Soner, NT & Zhang '14 and Possamaï, Tan & Zhou '16

Nonlinearity

Assumptions on $F : \mathbb{R}_+ \times \omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d_+ \longrightarrow \mathbb{R}$

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