Continuous time Principal Agent and optimal planning

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(Static) Principal-Agent Problem

- Principal delegates management of output process $X$, only observes $X$

- Agent devotes effort $a \implies X^a$, chooses optimal effort by
  \[
  V_A := \max_a \mathbb{E} U_A(\cdot - c(a))
  \]
(Static) Principal-Agent Problem

- Principal delegates management of output process $X$, only observes $X$
  pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \mapsto X^a$, chooses optimal effort by

  $$V_A(\xi) := \max_a \mathbb{E} \left[ U_A(\xi(X^a) - c(a)) \right] \implies \hat{a}(\xi)$$

- Principal chooses optimal contract by solving

  $$\max_\xi \mathbb{E} U_P (X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})) \quad \text{under constraint} \quad V_A(\xi) \geq \rho$$

  $\implies$ Non-zero sum Stackelberg game
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- Principal delegates management of output process $X$, only observes $X$
  pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \rightarrow X^a$, chooses optimal effort by

$$V_A(\xi) := \max_a \mathbb{E} \left[ U_A(\xi(X^a) - c(a)) \right] \quad \Rightarrow \quad \hat{a}(\xi)$$

- Principal chooses optimal contract by solving

$$\max_{\xi} \mathbb{E} \left[ U_P(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)})) \right] \quad \text{under constraint} \quad V_A(\xi) \geq \rho$$

$\Rightarrow$ Non-zero sum Stackelberg game
(Static) Principal-Agent Problem

- Principal delegates management of output process $X$, only observes $X$ and pays salary defined by contract $\xi(X)$.

- Agent devotes effort $a \Rightarrow X^a$, chooses optimal effort by
  \[ V_A(\xi) := \max_a \mathbb{E} \left[ U_A(\xi(X^a) - c(a)) \right] \Rightarrow \hat{a}(\xi) \]

- Principal chooses optimal contract by solving
  \[ \max_{\xi} \mathbb{E} \left[ U_P(X^{\hat{a}(\xi)} - \xi(X^{\hat{a}(\xi)}) \right] \text{ under constraint } V_A(\xi) \geq \rho \]

\[ \Rightarrow \text{Non-zero sum Stackelberg game} \]
(Static) Principal-Agent Problem $\implies$ Continuous time

- Principal delegates management of output process $X$, only observes $X$ and pays salary defined by contract $\xi(X)$

- Agent devotes effort $a \implies X^a$, chooses optimal effort by
  $$V_A(\xi) := \max_a \mathbb{E} U_A(\xi(X^a) - c(a)) \implies \hat{a}(\xi)$$

- Principal chooses optimal contract by solving
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$\implies$ Non-zero sum Stackelberg game
**Principal-Agent problem formulation**

**Contract** \( C = (\tau, \pi, \xi) \)

- \( \tau \) \( \mathcal{F} \)-stopping time, \( \pi \) \( \mathcal{F} \)-adapted, and \( \xi \) \( \mathcal{F}_\tau \)-measurable

**Agent problem**

\[
V_0^A(C) := \sup_{P \in \mathcal{P}} \mathbb{E}^P \left[ \xi(X) + \int_0^\tau (\pi_t - c_t(\nu_t)) dt \right]
\]

\( P \in \mathcal{P} \): weak solution of Output process:

\[
dX_t = b_t(X, \nu_t)dt + \sigma_t(X, \nu_t)dW_t^P \quad \mathbb{P} - \text{a.s.}
\]

for some \( \nu \) valued in \( U \)

**Principal problem** choose among acceptable contracts

\[
\Xi_\rho := \{ C : V_0^A(C) \geq \rho \}
\]

best contract, given Agent’s optimal response \( \mathbb{P}^*(C) \)

\[
V_0^P := \sup_{C \in \Xi_\rho} \mathbb{E}^{\mathbb{P}^*(C)} \left[ U(\ell(X) - \xi(X) - \int_0^\tau \pi_t(X)dt) \right]
\]
Principal-Agent problem formulation

Contract $C = (\tau, \pi, \xi)$

$\tau$ $\mathcal{F}$-stopping time, $\pi$ $\mathcal{F}$-adapted, and $\xi$ $\mathcal{F}_\tau$-mble

Agent problem

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$$dX_t = b_t(X, \nu_t) dt + \sigma_t(X, \nu_t) dW_t^P \quad \mathbb{P} - a.s.$$ for some $\nu$ valued in $U$

Principal problem choose among acceptable contracts

$$\Xi_{\rho} := \{C : V_0^A(C) \geq \rho\}$$

best contract, given Agent’s optimal response $P^*(C)$

$$V_0^P := \sup_{C \in \Xi_{\rho}} \mathbb{E}^{P^*(C)} \left[ U(\ell(X) - \xi(X) - \int_0^\tau \pi_t(X) dt) \right]$$

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Principal-Agent problem formulation

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$\tau$ $\mathcal{F}$-stopping time, $\pi$ $\mathcal{F}$-adapted, and $\xi$ $\mathcal{F}_\tau$-measurable

Agent problem

$$V_0^A(C) := \sup_{P \in \mathcal{P}} \mathbb{E}^P \left[ \xi(X) + \int_0^\tau (\pi_t - c_t(\nu_t)) dt \right]$$

$P \in \mathcal{P}$: weak solution of Output process:

d$X_t = \sigma_t(X, \nu_t)(\lambda_t(X, \nu_t)dt + dW_t^P)$ $P$-a.s.

for some $\nu$ valued in $U$

Principal problem choose among acceptable contracts

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A general solution approach

- Path-dependent Hamiltonian for the Agent problem:

\[
H^\pi_t(\omega, z, \gamma) := \sup_{u \in U} \left\{ b_t(\omega, u) \cdot z + \frac{1}{2} \sigma_t \sigma^T_t(\omega, u) : \gamma + \pi_t(\omega) - c_t(\omega, u) \right\}
\]

- For \( Y_0 \in \mathbb{R}, Z, \Gamma \in \mathcal{FX} \), define \( \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P} \)

\[
Y^Z,\Gamma_t = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - H^\pi_s(X, Z_s, \Gamma_s)ds
\]

Proposition

\[
V_A(\tau, \pi, Y^Z,\Gamma_\tau) = Y_0. \text{ Moreover } \mathbb{P}^* \text{ is optimal iff}
\]

\[
\nu^*_t = \text{Argmax}_{u \in U} H^\pi_t(Z_t, \Gamma_t) = \hat{\nu}(Z_t, \Gamma_t)
\]

Proof classical verification argument in stochastic control
A general solution approach

- Path-dependent Hamiltonian for the Agent problem:

\[ H^\pi_t(\omega, z, \gamma) := \sup_{u \in U} \left\{ b_t(\omega, u) \cdot z + \frac{1}{2} \sigma_t \sigma_t^\top(\omega, u) : \gamma + \pi_t(\omega) - c_t(\omega, u) \right\} \]

- For \( Y_0 \in \mathbb{R}, Z, \Gamma \in \mathbb{F}^X \) - prog meas, define \( \mathbb{P} \)-a.s. for all \( \mathbb{P} \in \mathcal{P} \)

\[ Y^{Z, \Gamma}_t = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - H^\pi_s(X, Z_s, \Gamma_s) ds \]

**Proposition**

\[ V_A(\tau, \pi, Y^Z_\tau, \Gamma) = Y_0. \text{ Moreover } \mathbb{P}^* \text{ is optimal iff} \]

\[ \nu_t^* = \text{Argmax}_{u \in U} H^\pi_t(Z_t, \Gamma_t) = \hat{\nu}(Z_t, \Gamma_t) \]

**Proof** classical verification argument in stochastic control.
Principal problem restricted to revealing contracts

Dynamics of the pair $(X, Y)$ under “optimal response”

\[ dX_t = b_t(X, \hat{\nu}(Z_t, \Gamma_t)) \, dt + \sigma_t(X, \hat{\nu}(Z_t, \Gamma_t)) \, dW_t \]
\[ dY_t^{Z, \Gamma} = Z_t \cdot dX_t + \frac{1}{2} \Gamma_t : d\langle X \rangle_t - H_t^\pi(X, Z_t, \Gamma_t) \, dt \]

is a (1 state augmented) controlled SDE with controls $(\pi, Z, \Gamma)$

\[ \implies \text{Principal’s value function under revealing contracts :} \]

\[ V_P \geq V_0(X_0, Y_0) := \sup_{(\tau, \pi) \in \nu} \mathbb{E} \left[ U(\ell(X) - Y^Z, \Gamma - \int_0^\tau \pi_t dt) \right], \quad \forall \ Y_0 \geq \rho \]

where \( \nu := \left\{ (Z, \Gamma) : Z \in \mathbb{H}^2(\mathcal{P}) \text{ and } \mathcal{P}^*(Y^{Z, \Gamma}_T) \neq \emptyset \right\} \)
Reduction to standard control problem

Theorem

Assume $\mathcal{V} \neq \emptyset$. Then

$$V_0^P = \sup_{Y_0 \geq \rho} V_0(X_0, Y_0)$$

Given maximizer $Y_0^*$, the corresponding optimal controls $(\tau^*, \pi^*, Z^*, \Gamma^*)$ induce an optimal contract $C^* = (\tau^*, \pi^*, \xi^*)$ with

$$\xi^* = Y_0^* + \int_0^T Z_t^* \cdot dX_t + \frac{1}{2} \Gamma_t^* : d\langle X \rangle_t - H_t^{\pi^*}(X, Z_t^*, \Gamma_t^*)dt$$

Sannikov '08
Cvitanić, Possamaï & NT '15
Lin, Ren, NT & Yang '19
Examples of volatility control problems

- Portfolio optimization
  \[ dV_t = \theta_t \cdot dS_t \]

- Demand-Response programs in electricity retail market
  \[ dX_t = \alpha_t dt + \beta_t \cdot dW_t \]

Open to many extensions

- agent may also choose optimally to quit (Sannikov)
- many agents, many principals under competition (Possamaï & ...)
- Limited liability (Possamaï & Villeneuve)
Recall the subclass of contracts

\[ Y_t^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s) ds \]

\[ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P} \]

To prove the main result, it suffices to prove the representation

for all \( \xi \in \mathbb{R} \) \( \exists (Y_0, Z, \Gamma) \) s.t. \( \xi = Y_T^{Z,\Gamma} \), \( \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P} \)

OR, weaker sufficient condition:

for all \( \xi \in \mathbb{R} \) \( \exists (Y_0^n, Z^n, \Gamma^n) \) s.t. \( "Y_T^{Z^n,\Gamma^n} \rightarrow \xi" \)
Recall the subclass of contracts

\[ Y_t^{Z,G} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - H_s(X, Y_s^{Z,G}, Z_s, \Gamma_s)ds \]

\[ \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P} \]

To prove the main result, it suffices to prove the representation

\[ \text{for all } \xi \in \mathcal{H} \exists (Y_0, Z, \Gamma) \text{ s.t. } \xi = Y_T^{Z,G}, \mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P} \]

OR, weaker sufficient condition:

\[ \text{for all } \xi \in \mathcal{H} \exists (Y_0^n, Z^n, \Gamma^n) \text{ s.t. } Y_T^{Z^n,G^n} \rightarrow \xi \]
Recall the subclass of contracts

\[ Y_t^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s) ds \]

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OR, weaker sufficient condition:

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The uncontrolled volatility case

- Path-dependent Hamiltonian for the Agent problem:
  \[ H_t^{\pi}(\omega, z, \gamma) := \sup_{u \in U} \left\{ b_t(\omega, u) \cdot z + \frac{1}{2} \sigma_t \sigma_t^T(\omega) : \gamma + \pi_t(\omega) - c_t(\omega, u) \right\} \]

- For \( Y_0 \in \mathbb{R}, Z, \Gamma \in \mathbb{F}^X \) — prog meas, define
  \[ Y_t^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - H_s^{\pi}(X, Z_s, \Gamma_s) ds \]
  \( \mathbb{P} \)-a.s. for all \( \mathbb{P} \in \mathcal{P} \)
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\[ H^\pi_t(\omega, z, \gamma) := \frac{1}{2} \sigma_t \sigma_t^\top(\omega) : \gamma + \sup_{u \in U} \left\{ b_t(\omega, u) \cdot z + \pi_t(\omega) - c_t(\omega, u) \right\} \]

- For \( Y_0 \in \mathbb{R}, Z, \Gamma \mathbb{F}^X – \text{prog meas} \), define

\[
Y_{t}^{Z, \Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2} \Gamma_s : d\langle X \rangle_s - \frac{1}{2} \sigma_t \sigma_t^\top : \gamma - H^\pi_s(X, Z_s, 0) ds
\]

\( \mathbb{P} \)-a.s. for all \( \mathbb{P} \in \mathcal{P} \)
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- For \( Y_0 \in \mathbb{R}, Z, \Gamma \in \mathbb{P}^{X} \) – prog meas, define

\[ Y_{t}^{Z, \Gamma} = Y_0 + \int_{0}^{t} Z_s \cdot dX_s - H_s^{\pi}(X, Z_s, 0)ds \]

\( \mathbb{P} \)-a.s. for all \( \mathbb{P} \in \mathcal{P} \)
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\]

- For \( Y_0 \in \mathbb{R}, \ Z, \Gamma \in \mathbb{R}^X \), define

\[
Y_{t}^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s - H_{s}^\pi(X, Z_s, 0)ds
\]

\( \mathbb{P}_0 \)-a.s. for some \( \mathbb{P}_0 \in \mathcal{P} \)

(\( \mathcal{P} \) dominated set of measures by Girsanov)

Representation problem reduces to

\[
Y_{T}^{Z,0} = \xi, \quad \mathbb{P}_0 \text{ - a.s.} \quad \text{Backward SDE...} \]

Pardoux & Peng ’90, El Karoui, Peng & Quenez ’96
The uncontrolled volatility case

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\[
H_t^\pi(\omega, z, \gamma) := \frac{1}{2} \sigma_t \sigma_t^\top(\omega) : \gamma + \sup_{u \in U} \left\{ b_t(\omega, u) \cdot z + \pi_t(\omega) - c_t(\omega, u) \right\}
\]

- For \( Y_0 \in \mathbb{R}, Z, \Gamma \mathbb{F}^X - \text{prog meas}, \) define

\[
Y^{Z, \Gamma}_t = Y_0 + \int_0^t Z_s \cdot dX_s - H^\pi_s(X, Z_s, 0)ds
\]

\( \mathbb{P}_0 - \text{a.s. for some } \mathbb{P}_0 \in \mathcal{P} \) (\( \mathcal{P} \) dominated set of measures by Girsanov)

Representation problem reduces to

\[
Y^{Z, 0}_\tau = \xi, \quad \mathbb{P}_0 - \text{a.s.} \quad \text{Backward SDE...} \quad \square
\]

Pardoux & Peng ’90, El Karoui, Peng & Quenez ’96
The controlled volatility case

- $H_t(\omega, y, z, \gamma)$ non-decreasing and convex in $\gamma$, the

$$H_t(\omega, y, z, \gamma) = \sup_\sigma \left\{ \frac{1}{2} \sigma^2 : \gamma - H^*_t(\omega, y, z, \sigma) \right\}$$

- Let $\hat{\sigma}_t^2 := \frac{d\langle X \rangle}{dt}$, and introduce

$$k_t := H_t(Y_t, Z_t, \Gamma_t) - \frac{1}{2} \hat{\sigma}_t^2 : \Gamma_t + H^*_t(Y_t, Z_t, \hat{\sigma}_t) : \geq 0 \quad \text{and} \quad \inf_{P \in \mathcal{P}} k_t = 0$$

Then, required representation $\xi = Y^Z, \Gamma, P-q.s.$ is equivalent to

$$\xi = Y_0 + \int_0^\tau Z_t \cdot dX_t + H^*_t(Y_t, Z_t, \hat{\sigma}_t)dt - \int_0^\tau k_t dt, \quad P-q.s.$$ 

$\implies$ 2BSDE up to approximation of nondecreasing process $K = \int_0^\tau k_t dt...$
The controlled volatility case

- \( H_t(\omega, y, z, \gamma) \) non-decreasing and convex in \( \gamma \), the

\[
H_t(\omega, y, z, \gamma) = \sup_{\sigma} \left\{ \frac{1}{2} \sigma^2 : \gamma - H^*_t(\omega, y, z, \sigma) \right\}
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- Let \( \hat{\sigma}_t^2 := \frac{d\langle X \rangle}{dt} \), and introduce

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Then, required representation \( \xi = Y^Z, \Gamma, \mathcal{P} - \text{q.s.} \) is equivalent to

\[
\xi = Y_0 + \int_0^\tau Z_t \cdot dX_t + H^*_t(Y_t, Z_t, \hat{\sigma}_t) dt - \int_0^\tau k_t dt, \quad \mathcal{P} - \text{q.s.}
\]

\[\Rightarrow 2\text{BSDE up to approximation of nondecreasing process } K = \int_0^\cdot k_t dt\ldots\]
The controlled volatility case

- \( H_t(\omega, y, z, \gamma) \) non-decreasing and convex in \( \gamma \), the

\[
H_t(\omega, y, z, \gamma) = \sup_{\sigma} \left\{ \frac{1}{2} \sigma^2 : \gamma - H^*_t(\omega, y, z, \sigma) \right\}
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k_t := H_t(Y_t, Z_t, \Gamma_t) - \frac{1}{2} \hat{\sigma}_t^2 : \Gamma_t + H^*_t(Y_t, Z_t, \hat{\sigma}_t) : \geq 0 \quad \text{and} \quad \inf_{\mathcal{P} \in \mathcal{P}} k_t = 0
\]

Then, required representation \( \xi = Y^{Z, \Gamma}_t, \mathcal{P} \)-q.s. is equivalent to

\[
\xi = Y_0 + \int_0^T Z_t \cdot dX_t + H^*_t(Y_t, Z_t, \hat{\sigma}_t)dt - \int_0^T k_t dt, \quad \mathcal{P} \text{ – q.s.}
\]

\( \Rightarrow \) 2BSDE up to approximation of nondecreasing process \( K = \int_0^\tau k_t dt \)...
Wellposedness of random horizon 2nd order backward SDE

\[ Y_{t \land \tau} = \xi + \int_{t \land \tau}^{\tau} F_s(Y_s, Z_s, \hat{\sigma}_s) \, ds - \int_{t \land \tau}^{\tau} Z_s \cdot dX_s + \int_{t \land \tau}^{\tau} dK_s, \quad \mathcal{P} - \text{q.s.} \]

\[ K \text{ non-decreasing, and inf}_{\mathcal{P} \in \mathcal{P}} \mathbb{E}_\mathcal{P}^{\mathcal{P}} \left[ \int_{s \land \tau}^{t \land \tau} dK_r \right] = 0, \ s \leq t \]

**Theorem (Y. Lin, Z. Ren, NT & J. Yang '18)**

Assume \( \exists \ \rho > -\mu, \ q > 1 : \mathcal{E}^L \left[ |e^{\rho \tau} \xi|^q \right] + \mathcal{E}^L \left[ (\int_0^\tau |e^{\rho t} f_t|^2 \, ds)^{\frac{q}{2}} \right] < \infty \)

Then, Random horizon 2BSDE has a unique solution \((Y, Z)\) with

\[ Y \in D_{\eta, \tau}^p, \ Z \in H_{\eta, \tau}^p \quad \text{for all} \quad \eta \in (-\mu, \rho), \ p \in [1, q) \]

\[ \| Y \|_{D_{\eta, \tau}^p}^p := \mathcal{E}^L \left[ \sup_{t \leq \tau} |e^{\eta t} Y_t|^p \right], \ \| Z \|_{H_{\eta, \tau}^p}^{\frac{p}{2}} := \mathcal{E}^L \left[ \left( \int_0^\tau |e^{\eta t} \hat{\sigma}_t^T Z_t|^2 \, dt \right)^{\frac{p}{2}} \right] \]

Extends Soner, NT & Zhang '12 and Possamaï, Tan & Zhou '18
Closely connected to G-BSDE, Hu, Ji, Peng & Song '14
Consider a crowd of agents in MFG equilibrium:

\[ V_0^A(\mu, \xi) := \sup_{P \in \mathcal{P}} J(\mu, \xi, P) = J(\mu, \xi, \hat{P}_{\xi}^\mu) \]

where

\[ J(P, \mu, \xi) := \mathbb{E}^P \left[ \xi(X) - \int_0^T c_t(\mu_t, \alpha_t, \beta_t) \, dt \right] \]

and \( P \in \mathcal{P} \) is weak solution of controlled process:

\[ P \circ (X_0)^{-1} = \mu_0, \text{ and } dX_t = \sigma_t(X, \beta_t) \left[ \lambda_t(X, \alpha_t) \, dt + dW^P_t \right], \mathbb{P}-\text{a.s.} \]

**Definition (Mean field game equilibrium)**

\( \hat{\mu} \) is an MFG equilibrium if \( \hat{P}_{\xi}^\mu \circ (X_t)^{-1} = \hat{\mu}_t \), for all \( t \leq T \)
P.L. Lions’ Planning Problem

- uncontrolled diffusion $\sigma_t(\beta) = I_d$
- Markov setting: $\lambda_t(\omega) = \lambda_t(\omega_t)$, $c...$, and $\xi(\omega) = g(\omega_T)$

### Planning Problem

- Let $\mu_0, \nu$ be given probability measures on $\mathbb{R}^d$
- Find $g : \mathbb{R}^d \mapsto \mathbb{R}$ so that

  $\text{MFG equilibrium } \hat{\mu} \text{ exists and } \hat{\mu}_0 = \mu_0, \hat{\mu}_T = \nu$

### Interpretation: optimal transport, regulation

Start from crowd distributed as $\mu_0$. Choose an appropriate incentive cost $g : \mathbb{R}^d \mapsto \mathbb{R}$ so as to force the MFG equilibrium to the target distribution $\nu$ at time $T$.

$\Rightarrow$ Unique solution exists for any pair $(\mu, \nu)$...

Lions ’10, Achdou Y, Camilli F & Capuzzo Dolcetta ’12, Porretta ’14
Path-dependent formulation of the Planning Problem

Allow the incentive cost $\xi$ to be path-dependent

Path-dependent Planning Problem

• Let $\mu_0, \nu$ be given probability measures on $\mathbb{R}^d$

• Find $\xi : \Omega \rightarrow \mathbb{R}$, $\mathcal{F}_T$–measurable, so that

$$\text{MFG equilibrium } \hat{\mu} \text{ exists and } \hat{\mu}_0 = \mu_0, \quad \hat{\mu}_T = \nu$$

• More freedom for the choice of incentive regulation

• Multiple solutions, in general
MFG equilibria with varying path-dependent cost \( \xi \)

**Forward description of MFG equilibria**

For all controls \((Z, \Gamma)\), let \( \xi^{Z,\Gamma} := Y^{Z,\Gamma}_T \), where \( Y^{Z,\Gamma} \) is defined by the McKean-Vlasov controlled process

\[
dY^{Z,\Gamma}_t = Z_t \cdot dX_t + \frac{1}{2} \Gamma_t : d\langle X \rangle_t - H_t(Y^{Z,\Gamma}_t, Z_t, \Gamma_t, \mu_t)
\]

\[
\mu_t = P \circ X_t^{-1} \text{ distribution of } X_t \text{ with controls defined as maximizers of } H
\]

\[
dX_t = \nabla_z H_t(\cdots) dt + \left[2\nabla_\gamma H_t(\cdots)\right]^{1/2} dW_t
\]

- Multiple solutions, in general
- Skorohod embedding problem is a particular case:
  - planning exists iff \( \mu_0 \preceq \nu \) in convex order
  - Many solutions exist... sometimes corresponding to various optimization criteria!
Optimal Planning Problem

MFG transport plans from $\mu_0$ to $\nu$ in $\text{Prob}(\mathbb{R}^d)$

- $\Xi(\mu_0, \nu) : \xi \in \Lambda^0(\Omega, \mathcal{F}_T)$ s.t. there exists an MFG equilibrium $\mu$ satisfying $\hat{P}_\xi \circ (X_T)^{-1} = \nu$
- Given the planner criterion $\phi : \Omega \times \Lambda^0(\Omega, \mathcal{F}_T) \rightarrow \mathbb{R}$, solve

$$V^P := \sup_{\xi \in \Xi(\mu_0, \nu)} \mathbb{E}[\hat{P}_\mu^\xi \phi(X, \xi(X))]$$

Theorem (Z. Ren, X. Tan & NT)

Planner problem can be restricted to forward MV transport plans

$$V^P = \sup_{Z, \Gamma : \hat{P}_{Z, \Gamma} \circ (X_T)^{-1} = \nu} \mathbb{E}[\hat{P}_{Z, \Gamma}^Z \phi(X, Y_{T}^{Z, \Gamma})]$$
BON ANNIVERSAIRE NICOLE
Nonlinear expectation operators

\( \mathcal{P}^0 \) : subset of local martingale measures, i.e.

\[ dX_t = \sigma_t dW_t, \quad \mathbb{P} \text{ – a.s. for all } \mathbb{P} \in \mathcal{P}^0 \]

\( \Rightarrow \) Nonlinear expectation

\[ \mathcal{E} := \sup_{\mathbb{P} \in \mathcal{P}^0} \mathbb{E}^{\mathbb{P}} \]

Similarly, \( \mathcal{P}^L \) : subset of measures \( \mathcal{Q}^\lambda \) such that

\[ dX_t = \sigma_t (\lambda_t dt + dW_t), \quad \mathbb{Q} \text{ – a.s. for some } \lambda, \mathbb{F} \text{ – adapted, } |\lambda| \leq L \]

\( \Rightarrow \) Another nonlinear expectation

\[ \mathcal{E}^L := \sup_{\mathbb{P} \in \mathcal{P}^L} \mathbb{E}^{\mathbb{Q}} \]

\( \mathcal{E} \) and \( \mathcal{E}^L \) will play the role of Sobolev norms...
Nonlinearity

Assumptions on $F : \mathbb{R}_+ \times \mathcal{C} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_+^d \rightarrow \mathbb{R}$

(C1$_L$) Lipschitz in $(y, \sigma z)$:

$$|F(., y, z, \sigma) - F(., y', z', \sigma)| \leq L \left( |y - y'| + |\sigma(z - z')| \right)$$

(C2$_\mu$) Monotone in $y$:

$$(y - y') \cdot \left[ F(., y, .) - F(., y', .) \right] \leq -\mu |y - y'|^2$$

Denote $f^0_t := F_t(0, 0, \hat{\sigma}_t)$

Remark Deterministic finite horizon $\tau = T$ : (C2)$_\mu$ not needed

Soner, NT & Zhang ’14 and Possamaï, Tan & Zhou ’16
Nonlinearity

Assumptions on \( F : \mathbb{R}_+ \times \omega \times \mathbb{R} \times \mathbb{R}^d \times S^d \rightarrow \mathbb{R} \)

\((C1_L)\) Lipschitz in \((y, \sigma z)\):

\[ |F(., y, z, \sigma) - F(., y', z', \sigma)| \leq L \left( |y - y'| + |\sigma(z - z')| \right) \]

\((C2_\mu)\) Monotone in \(y\):

\[ (y - y') \cdot [F(., y, .) - F(., y', .)] \leq -\mu |y - y'|^2 \]

Denote \( f^0_t := F_t(0, 0, \hat{\sigma}_t) \)

**Remark** Deterministic finite horizon \( \tau = T \) : \((C2)_\mu\) not needed

Soner, NT & Zhang ’14 and Possamaï, Tan & Zhou ’16