McKean-Vlasov (or MF) backward-forward stochastic differential equations and MK-V nonzero sum stochastic differential games

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Outlines

1. The Linear-Quadratic nonzero-sum differential game. Link with Forward-Backward SDEs ;
2. Forward-Backward McKean-Vlasov SDEs ;
3. Existence of a Nash equilibrium for the NZSD game.

1. The MKV LQ nonzero-sum differential game

Consider a system controlled by m agents or players whose dynamics is: $X^u := (X_t^u)_{t\le T}$ solution of:

$$X_t^u = x + \int_0^t \{A_s X_s^u + \sum_{k=1,m} C_s^k u_s^k + D_s \mathbb{E}[X_s^u] + \beta_s\} ds \\ + \int_0^t \{\sigma_s X_s^u + \alpha_s\} dW_s, \ t \le T, \tag{1}$$

i) The player i acts with a control $u^i := (u_t^i)_{t\le T}$, valued in $\mathbb{R}^{p_i}$ adapted and $dt \times d\mathbb{P}$-square integrable. The set $\mathcal{U}^i$ is of those controls (admissible controls).

ii) $A = (A_t)_{t \leq T}$, $\beta = (\beta_t)_{t \leq T}$, $\alpha = (\alpha_t)_{t \leq T}$, $C^k = (C^k_t)_{t \leq T}$ and $\sigma = (\sigma_t)_{t \leq T}$ are bounded and adapted stochastic processes with appropriate dimensions ; $D = (D_t)_{t \leq T}$ is deterministic.

When a collective strategy $u = (u^i)_{i=1,m} \in \mathcal{U}$ is implemented, the payoff of player i is $J_i(u)$ given by:

$$J_i(u) = J_i((u^i)_{i=1,m}) := \frac{1}{2}\mathbb{E}[(X_T^u)^\top Q_i X_T^u +$$

$$\int_0^T \{(X_s^u)^\top . M_s^i . X_s^u + u_s^\top . N_s^i . u_s + \mathbb{E}[X_s^u]^\top . \Gamma_s^i . \mathbb{E}[X_s^u]\} ds]$$

where:
i) for any $i = 1, m$, $M^i = (M_t^i)_{t \le T}$ and $N^i = (N_t^i)_{t \le T}$ are adapted processes valued respectively in $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{p_i \times p_i}$
ii) $\Gamma^i = (\Gamma_t^i)_{t \le T}$ is deterministic and valued in $\mathbb{R}^{n \times n}$.

They are all positive.

The problem is then to find a Nash equilibrium point for the game, i.e., a collective control $u^* = (u_1^*, ..., u_n^*)$ such that for any $i = 1, m$,

$$J_i(u_1^*, ..., u_m^*) \leq J_i(u_1^*, ..., u_{i-1}^*, u_i, u_{i+1}^*, ..., u_m^*), \ \forall u_i \in \mathcal{U}^i.$$

Remark: If $m = 2$ and $J_1 + J_2 = 0$, the game is of zero-sum type and NEP is just a saddle-point.

For $i = 1, m$, let H_i be the Hamiltonian associated with the i-th player which is:

$$H_i(t, \omega, z_i, x, u_1, ..., u_m, \zeta) := z_i^\top \{A_t(\omega)x + \sum_{k=1,m} C_t^k(\omega)u^k +$$

$$D_t \,.\, \zeta + \beta_t(\omega)\} + \frac{1}{2}\{x^\top M_t^i(\omega)x + u_i^\top N_t^i(\omega)u_i + \zeta^\top \Gamma_t^i \zeta\}$$

with $u^i \in \mathbb{R}^{p_i}$, $z^i \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^n$.

Next for $i = 1, m$, let $\tilde{u}^i$ be the functions defined by:

$$\tilde{u}^i(t, \omega, z^i) := -(N_t^i)^{-1}(C_t^i)^\top z_i, \ t \leq T.$$

The measurable functions $\tilde{u}^i$, $i = 1, m$, verify: $\forall i = 1, m$,

$$H_i(t, \omega, z_i, x, (\tilde{u}^j(t, \omega, z^j))_{j=1,m}, \zeta) \leq$$

$$H_i(t, \omega, z_i, x, \tilde{u}^1(t, \omega, z^1), ..., \tilde{u}^{i-1}(t, \omega, z^{i-1}), u^i,$$

$$\tilde{u}^{i+1}(t, \omega, z^{i+1}), ..., \tilde{u}^m(t, \omega, z^m), \zeta), \ \forall u^i \in \mathbb{R}^{p_i}.$$

Proposition

Assume there exist $Prog$-meas. proc. $(X,(p^1,q^1),...,(p^m,q^m))$ which belong to $\mathcal{M}^2$ and which solve the following Backward-Forward stochastic differential equation: $\forall t \leq T$,

$$\begin{cases} X_t = x + \int_0^t \{A_s X_s + \sum_{k=1,m} C_s^k \tilde{u}^k(s,p_s^k) + D_s \mathbb{E}[X_s] + \beta_s\} ds \\ \qquad\qquad + \int_0^t \{\sigma_s X_s + \alpha_s\} dW_s \\ p_t^i = Q^i X_T + \int_t^T \{A_s^\top p_s^i + M_s^i X_s + \Gamma_s^i \mathbb{E}[X_s] \\ \qquad + D_s \mathbb{E}[p_s^i] + \sum_{j=1,m} (\sigma_s^j)^\top q_s^{i,j}\} ds - \int_t^T q_s^i dW_s, \quad i = 1, m. \end{cases} \tag{2}$$

Then the control $\tilde{u} := (\tilde{u}^j)_{j=1,m} = ((\tilde{u}^j(t,\omega,p^j))_{t\leq T})_{j=1,m}$ is an open-loop Nash equilibrium point for the McKean-Vlasov NZSDG.

Main steps of the proof: First note that

$$\begin{aligned}
\theta_1^\top \Sigma \theta_1 - \theta_2^\top \Sigma \theta_2 &= (\theta_1 - \theta_2)^\top \Sigma (\theta_1 - \theta_2) + 2(\theta_1 - \theta_2)^\top \Sigma \theta_2 \\
&\geq 2(\theta_1 - \theta_2)^\top \Sigma \theta_2
\end{aligned}$$

if Σ is positive.

Take $i = 1$. i) For $v \in \mathcal{U}^1$,

$$J^1(v, \tilde{u}^2, \ldots, \tilde{u}^m) - J^1(\tilde{u}^1, \ldots, \tilde{u}^m)$$

$$= \tfrac{1}{2}\mathbb{E}[\{(X_T^v)^\top Q_1 X_T^v - (X_T)^\top Q_1 X_T\}$$

$$+ \int_0^T \{(X_s^v)^\top . M_s^1 . X_s^v + v_s^\top . N_s^1 . v_s + \mathbb{E}[X_s^v]^\top . \Gamma_s^1 . \mathbb{E}[X_s^v]\} ds$$

$$- \int_0^T \{(X_s)^\top . M_s^1 . X_s + \tilde{u}_s^\top . N_s^1 . \tilde{u}_s + \mathbb{E}[X_s]^\top . \Gamma_s^1 . \mathbb{E}[X_s]\} ds]$$

But the matrices M^1, N^1, Γ^1 and Q^1 are positive. Then:

$$J^1(v, \tilde{u}^2, \ldots, \tilde{u}^m) - J^1(\tilde{u}^1, \ldots, \tilde{u}^m) \geq$$

$$\mathbb{E}[(X_T^v - X_T)^\top Q^1 X_T + \int_0^T \{(X_s^v - X_s)^\top M_s^1 X_s + (v_s - \tilde{u}_s^1)^\top N_s^1 \tilde{u}_s^1 +$$

$$(\mathbb{E}[X_s^v - X_s])^\top \Gamma_s^1 \mathbb{E}[X_s]\} ds].$$

As $p_T^1 = Q^1 X_T$ then by Itô's formula and expectation:

$$\mathbb{E}[X_T Q^1 (X_T^v - X_T)] = \mathbb{E}[p_T^1 (X_T^v - X_T)]$$

$$= \mathbb{E}[\int_0^T \{p_s^1 D_s \mathbb{E}[(X_s^v - X_s)] - X_s M_s^1 (X_s^v - X_s)$$

$$+(v_s - \tilde{u}^1(s, p_s^1)) C_s^{1\top} p_s^1 - (X_s^v - X_s)\mathbb{E}[\Gamma_s^1 X_s]$$

$$-(X_s^v - X_s) D_s \mathbb{E}[p_s^1]\} ds]$$

which yields,

$$J^1(v,\tilde{u}^2,\ldots,\tilde{u}^m) - J^1(\tilde{u}^1,\ldots,\tilde{u}^m)$$

$$\geq \mathbb{E}[\int_0^T (v_s - \tilde{u}_s^1)^\top \{N_s^1 \tilde{u}_s(s,p_s^1) + C_s^1 p_s^1\} ds] = 0.$$

C. The Backward-Forward equations of McKean-Vlasov type

We consider the following BFSDEs of McKean-Vlasov type: $t \leq T$

$$\begin{cases} X_t = x + \int_0^t f(s, X_s, Y_s, Z_s, \mathbb{P}_{(X_s,Y_s)})ds + \\ \qquad\qquad\qquad \int_0^t \sigma(s, X_s, Y_s, Z_s, \mathbb{P}_{(X_s,Y_s)})dW_s, \\ Y_t = g(X_T, \mathbb{P}_{X_T}) - \int_t^T h(s, X_s, Y_s, Z_s, \mathbb{P}_{(X_s,Y_s)})ds - \int_t^T Z_s dW_s, \end{cases} \tag{3}$$

i) X and Y have the same dimension.

ii) The functions f, σ, g and h are assumed to be Lipschitz continuous in the variables x, y, z and ν.

$M_2(\mathbb{R}^k)$:= the set of probability measures on $\mathbb{R}^k$ with moments of order 2.
For $\mu_1, \mu_2 \in \mathcal{M}_2(\mathbb{R}^k)$, the 2-Wasserstein distance is

$$d(\mu_1, \mu_2) := \inf \left\{ \left(\int_{\mathbb{R}^k \times \mathbb{R}^k} |x - y|^2 F(dx, dy) \right)^{1/2} \right\} \tag{4}$$

over $F \in \mathbb{M}(\mathbb{R}^k \times \mathbb{R}^k)$ with marginals μ_1 and μ_2.

In terms of a coupling between two square-integrable random variables ξ and ξ' defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$d(\mu, \nu) = \inf \left\{ \left(\mathbb{E}\left[|\xi - \xi'|^2 \right] \right)^{1/2}, \ \text{law}(\xi) = \mu_1, \, \text{law}(\xi') = \mu_2 \right\}. \tag{5}$$

Then we have:

$$d^2(\mathbb{P}_\xi, \mathbb{P}_{\bar{\xi}},) \leq \mathbb{E}[|\xi - \bar{\xi}|^2], \tag{6}$$

where $\mathbb{P}_\xi :=$ law(ξ) and $\mathbb{P}_{\xi'} :=$ law(ξ').

For $t \in [0,T], \nu \in \mathbb{M}_2(\mathbb{R}^m \times \mathbb{R}^m), u = (x,y,z)$ and $u' = (x',y',z')$ in $\mathbb{R}^{m+m+m\times m}$, we define the function $\mathcal{A}(t,u,u',\nu)$ by

$$\begin{aligned}\mathcal{A}(t,u,u',\nu) = & (f(s,x,y,z,\nu) - f(s,x',y',z',\nu)) \cdot (y-y') \\ & +(h(s,x,y,z,\nu) - h(s,x',y',z',\nu)) \cdot (x-x') \\ & +[\sigma(s,x,y,z,\nu) - \sigma(s,x',y',z',\nu), z-z'].\end{aligned} \tag{7}$$

We consider the following assumptions:

$$
(H1)\begin{cases}
\text{(i) there exists } k>0, \text{ s.t. for all } t\in[0,T], \\
\nu\in\mathbb{M}_2(\mathbb{R}^m\times\mathbb{R}^m), u=(x,y,z), u'\in\mathbb{R}^{m+m+m\times m}, \\
\qquad \mathcal{A}(t,u,u',\nu)\leq -k|x-x'|^2, \quad \mathbb{P}\text{-a.s.} \\
\text{(ii) there exists } k'>0, \text{ s.t.} \\
(g(x,\nu)-g(x',\nu))\cdot(x-x')\geq k'|x-x'|^2, \quad \mathbb{P}\text{-a.s.}
\end{cases}
$$

Theorem

Under (H1) and if the Lipschitz constant of f, g, h and σ w.r.t ν are small enough then there exists a unique process $U = (X, Y, Z)$ which belongs to $\mathcal{M}^{2,m+m+m\times m}$ and which solves the FBSDE (3).

When the BFSDE does not depend on the laws, Hu-Peng ('95), Ham. ('98), Peng-Wu ('99), etc.

$a)$ Uniqueness of the solution

Let $U' = (X', Y', Z')$ be another solution to (3). Set $\nu_s = \mathbb{P}_{(X_s, Y_s)}$, $\nu'_s = \mathbb{P}_{(X'_s, Y'_s)}$ and

$$\Gamma_T = E[(X'_T - X_T) \cdot (Y'_T - Y_T)]$$

By Itô's formula,

$$\Gamma_T = \mathbb{E}\left[\int_0^T \{(f(s,U'_s,\nu'_s) - f(s,U_s,\nu_s)) \cdot (Y'_s - Y_s) \right.$$
$$+(h(s,U'_s,\nu'_s) - h(s,U_s,\nu_s)) \cdot (X'_s - X_s)$$
$$+[\sigma(s,U'_s,\nu'_s) - \sigma(s,U_s,\nu_s), Z'_s - Z_s]\}\, ds] \tag{8}$$

$$\Gamma_T = \mathbb{E}[(X'_T - X_T) \cdot (Y'_T - Y_T)]$$

$$= \mathbb{E}[(X'_T - X_T)(g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})]$$

$$= \mathbb{E}[(X'_T - X_T)\{(g(X'_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X'_T}))$$

$$+(g(X_T, \mathbb{P}_{X'_T}) - g(X_T, \mathbb{P}_{X_T})\}]$$

In view of (H1), (6) and the Lipschitz continuity of g, we have

$$\begin{aligned}\Gamma_T &\geq k' E[|X'_T - X_T|^2] - C^g_\nu E[|X'_T - X_T|] d(\mu'_T, \mu_T) \\ &\geq k' E[|X'_T - X_T|^2] - C^g_\nu E[|X'_T - X_T|] E[|X'_T - X_T|^2]^{1/2} \\ &\geq (k' - C^g_\nu) E[|X'_T - X_T|^2].\end{aligned}$$

On the other hand, since f, h, σ are Lipschitz in ν, then

$$\Gamma_T \leq \mathbb{E}[\int_0^T \{\mathcal{A}(s, U_s, U'_s, \nu_s) +$$

$$C_\nu(|X'_s - X_s| + |Y'_s - Y_s| + \|Z'_s - Z_s\|) d(\nu_s, \nu'_s)\} ds].$$

As by (6),

$$d^2(\nu_s, \nu'_s) \leq \mathbb{E}[|X'_s - X_s|^2 + |Y'_s - Y_s|^2] \leq \mathbb{E}[\|U'_s - U_s\|^2],$$

we obtain

$$\Gamma_T \leq \mathbb{E}\left[\int_0^T \left\{-k|X'_s - X_s|^2 + C_{\nu,T}\|U'_s - U_s\|^2\right\} ds\right].$$

But

$$\mathbb{E}[\int_0^T \|U_s' - U_s\|^2 ds] \leq \tilde{C}\mathbb{E}[|X_T' - X_T|^2 + \int_0^T |X_s' - X_s|^2 ds].$$

Then

$$\Gamma_T \leq C_{\nu,T}\tilde{C}\mathbb{E}[|X'_T - X_T|^2] + (C_{\nu,T}\tilde{C} - k)\mathbb{E}\left[\int_0^T |X'_s - X_s|^2 ds\right].$$

Therefore,

$$(k' - C^g_\nu - C_{\nu,T}\tilde{C}))\mathbb{E}[|X'_T - X_T|^2] + (k - C_{\nu,T}\tilde{C}))\mathbb{E}\left[\int_0^T |X'_s - X_s|^2 ds\right] \leq 0.$$

Now since C^g_ν and $C_{\nu,T}$ are small enough then $X = X'$ and finally it holds that $Y = Y'$ and $Z = Z'$, $dt \otimes d\mathbb{P}\text{-a.s.}$

$b)$ Existence of a solution.

Let $\delta > 0$ and $(X^n, Y^n, Z^n)_{n\geq 0}$ defined recursively as follows. $(X^0, Y^0, Z^0) = (0, 0, 0)$ and, for $n \geq 1, U^n = (X^n, Y^n, Z^n)$ satisfies

$$\begin{aligned} X_t^{n+1} &= x + \int_0^t \left\{ f(s, U_s^{n+1}, \nu_s^n) - \delta(Y_s^{n+1} - Y_s^n) \right\} ds \\ &\qquad + \int_0^t \left\{ \sigma(s, U_s^{n+1}, \nu_s^n) - \delta(Z_s^{n+1} - Z_s^n) \right\} dW_s, \\ Y_t^{n+1} &= g(X_T^{n+1}, \mu_T^n) - \int_t^T h(s, U_s^{n+1}, \nu_s^n) ds - \int_t^T Z_s^{n+1} dW_s, \end{aligned} \tag{9}$$

where $\nu_t^n := \mathbb{P}_{(X_t^n, Y_t^n)}$ and $\mu_T^n := \mathbb{P}_{X_T^n}$.

The solution of (9) exists and is unique under (H1) (Hu-Peng '95, Ham. '98, Peng-Wu '99).

It is enough to show that $(U^n)_{n\geq 0}$ and $(X_T^n)_{n\geq 0}$ are Cauchy sequences.
For $n \geq 1,\ t \in [0,T]$, set

$$\hat{X}_t^{n+1} := X_t^{n+1} - X_t^n,\ \ \hat{Y}_t^{n+1} := Y_t^{n+1} - Y_t^n,\ \ \hat{Z}_t^{n+1} := Z_t^{n+1} - Z_t^n$$

and for $\varphi \in \{f, h, \sigma\}$,

$$\hat{\varphi}^{n+1}(t) := \varphi(t, U_t^{n+1}, \nu_t^n) - \varphi(t, U_t^n, \nu_t^n)$$

and

$$\bar{\varphi}^n(t) := \varphi(t, U_t^n, \nu_t^n) - \varphi(t, U_t^n, \nu_t^{n-1}).$$

By Itô's formula and expectation, we obtain:

$$\begin{aligned}
&\mathbb{E}[\hat{X}_T^{n+1} \cdot (g(X_T^{n+1}, \mu_T^n) - g(X_T^n, \mu_T^{n-1}))] \\
&\quad + \delta \mathbb{E}\left[\int_0^T \left(|\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2\right) ds\right] \\
&\quad - \mathbb{E}\left[\int_0^T \left(\hat{X}_s^{n+1} \cdot \hat{h}^{n+1}(s) + \hat{Y}_s^{n+1} \cdot \hat{f}^{n+1}(s) + [\hat{\sigma}^{n+1}(s), \hat{Z}_s^{n+1}]\right) ds\right] \\
&= \delta \mathbb{E}\left[\int_0^T \left(\hat{Y}_s^{n+1} \cdot \hat{Y}_s^n + [\hat{Z}_s^{n+1}, \hat{Z}_s^n]\right) ds\right].
\end{aligned} \tag{10}$$

Using the Lipschitz continuity of g, Young's inequality, (6) and (H1(ii)), we have

$$
\begin{aligned}
&\mathbb{E}[\hat{X}_T^{n+1}\cdot(g(X_T^{n+1},\mu_T^n)-g(X_T^n,\mu_T^{n-1}))]\\
&\quad=\mathbb{E}[\hat{X}_T^{n+1}\cdot(g(X_T^{n+1},\mu_T^n)-g(X_T^n,\mu_T^n))]\\
&\qquad\qquad+\mathbb{E}[\hat{X}_T^{n+1}\cdot(g(X_T^n,\mu_T^n)-g(X_T^n,\mu_T^{n-1}))]\\
&\quad\geq k'\mathbb{E}[|\hat{X}_T^{n+1}|^2]-C_\nu^g\mathbb{E}[|\hat{X}_T^{n+1}|]d(\mu_T^n,\mu_T^{n-1})\\
&\quad\geq k'\mathbb{E}[|\hat{X}_T^{n+1}|^2]-\tfrac{C_\nu^g\varepsilon}{2}\mathbb{E}[|\hat{X}_T^{n+1}|^2]-\tfrac{C_\nu^g}{2\varepsilon}d^2(\mu_T^n,\mu_T^{n-1})\\
&\quad\geq (k'-\tfrac{C_\nu^g\varepsilon}{2})\mathbb{E}[|\hat{X}_T^{n+1}|^2]-\tfrac{C_\nu^g}{2\varepsilon}\mathbb{E}[|\hat{X}_T^n|^2].
\end{aligned}
$$

Again, by the Lipschitz continuity of f, h, σ, Young's inequality, (6) and (H1(i)), we also have

$$\hat{X}_t^{n+1} \cdot \hat{h}^{n+1}(t) + \hat{Y}_t^{n+1} \cdot \hat{f}^{n+1}(t) + [\hat{\sigma}^{n+1}(t), \hat{Z}_t^{n+1}]$$

$$= \mathcal{A}(t, U_t^{n+1}, U_t^n, \nu_t^n) + \hat{X}_t^{n+1} \cdot \bar{h}^n(t)$$

$$+\hat{Y}_t^{n+1} \cdot \bar{f}^n(t) + [\bar{\sigma}^n(t), \hat{Z}_t^{n+1}]$$

Then

$$\mathbb{E}\left[\int_0^T \left(\hat{X}_s^{n+1}\cdot\hat{h}^{n+1}(s)+\hat{Y}_s^{n+1}\cdot\hat{f}^{n+1}(s)+[\hat{\sigma}^{n+1}(s),\hat{Z}_s^{n+1}]\right)ds\right]$$

$$\leq \mathbb{E}[\int_0^T\{(\frac{C_\nu\alpha}{2}-k)|\hat{X}_t^{n+1}|^2+$$

$$\frac{C_\nu\alpha}{2}(|\hat{Y}_t^{n+1}|^2+\|\hat{Z}_t^{n+1}\|^2)+\frac{3C_\nu}{2\alpha}(|\hat{X}_t^{n}|^2+|\hat{Y}_t^{n}|^2)\}ds].$$

Furthermore, we have

$$\mathbb{E}\left[\int_0^T \left(\hat{Y}_s^{n+1}\cdot\hat{Y}_s^{n} + [\hat{Z}_s^{n+1},\hat{Z}_s^{n}]\right) ds\right]$$

$$\leq \frac{1}{2}\mathbb{E}\left[\int_0^T \left(|\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2 + |\hat{Y}_s^{n}|^2 + \|\hat{Z}_s^{n}\|^2\right) ds\right].$$

Those inequalities lead to:

$$(k' - \frac{C_\nu^g \varepsilon}{2})\mathbb{E}[|\hat{X}_T^{n+1}|^2]$$

$$+\mathbb{E}\left[\int_0^T \left\{(k - \frac{C_\nu \alpha}{2})|\hat{X}_s^{n+1}|^2 + (\frac{\delta}{2} - \frac{C_\nu \alpha}{2})(|\hat{Y}_s^{n+1}|^2 + \|\hat{Z}_s^{n+1}\|^2)\right\} ds\right]$$

$$\leq \frac{C_\nu^g}{2\varepsilon}\mathbb{E}[|\hat{X}_T^n|^2] + \mathbb{E}\left[\int_0^T \left\{\frac{3C_\nu}{2\alpha}|\hat{X}_s^n| + (\frac{3C_\nu}{2\alpha} + \frac{\delta}{2})|\hat{Y}_s^n|^2 + \frac{\delta}{2}\|\hat{Z}_s^n\|^2\right\} ds\right].$$

Setting

$$\gamma := \min\{k' - \frac{C_\nu^g \varepsilon}{2}, k - \frac{C_\nu \alpha}{2}, \frac{\delta}{2} - \frac{C_\nu \alpha}{2}\}, \quad \theta := \frac{C_\nu^g}{2\varepsilon} + \frac{3C_\nu}{\alpha} + \frac{\delta}{2},$$

we obtain

$$\mathbb{E}[|\hat{X}_T^{n+1}|^2] + E\left[\int_0^T \|\hat{U}_s^{n+1}\|^2 ds\right] \leq \frac{\theta}{\gamma}\left(\mathbb{E}[|\hat{X}_T^n|^2] + \mathbb{E}\left[\int_0^T \|\hat{U}_s^n|^2\| ds\right]\right).$$

Choose α, ε and δ so that $\theta < \gamma$, then $(X_T^n)_{n\geq 0}$ is a Cauchy sequence and $(X^n)_{n\geq 0}, (Y^n)_{n\geq 0}$ and $(Z^n)_{n\geq 0}$ are Cauchy sequences. Hence, if X, Y and Z are the respective limits then passing to the limit in (9), yields (X, Y, Z) is a solution of (3).

Other assumptions

One can consider instead of (H1) the following assumptions (H2):

$$
(H2)\begin{cases}
\text{(i) there exists } k>0, \text{ s.t. for all } t\in[0,T], \\
\qquad \nu \in \mathbb{M}_2(\mathbb{R}^m\times\mathbb{R}^m), u, u' \in \mathbb{R}^{m+m+m\times m}, \\
\\
\mathcal{A}(t,u,u',\nu) \le -k(|y-y'|^2 + \|z-z'\|^2), \quad \mathbb{P}\text{-a.s.} \\
\\
\text{(ii) there exists } k'>0, \text{ s.t. for all } \nu \in \mathbb{M}_2(\mathbb{R}^m\times\mathbb{R}^m), \\
x, x' \in \mathbb{R}^m, \\
\\
(g(x,\nu)-g(x',\nu))\cdot(x-x') \ge k'|x-x'|^2, \quad \mathbb{P}\text{-a.s.} \ .
\end{cases}
$$

D. Existence of an open-loop Nash equilibrium point

Let us consider the following assumptions on the data of the NZSDG:

Assumptions (A):

i) For any $i = 1, ..., m$, the process $(C_t^i (N_t^i)^{-1} (C_t^i)^\top)_{t \le T}$ is deterministic and independant of t. We set

$$K^i := C_t^i (N_t^i)^{-1} (C_t^i)^\top.$$

ii) There exists a constant $\gamma > 0$ such that

$$x^\top (\sum_{i=1,m} K^i Q^i) x \ge \gamma |x|^2 \text{ and } x^\top (\sum_{i=1,m} K^i M_t^i) x \ge \gamma |x|^2, \; t \le T.$$

iii) For any $i = 1, m$, $K^i.A_t^\top = A_t^\top.K^i$, $K^i.D_t = D_t.K^i$ and $K^i.\sigma_t^{j\top} = \sigma_t^{j\top}.K^i$.

iv) D and $\sum_{i=1,m} K^i \Gamma_s^i$ are small enough.

Remark: Those assumptions are easy to verify when the coefficients are constant are in dimension 1.

Proposition

Under (A), the FBSDE (2) *associated with the NZSDG has a unique solution.*

Actually let us consider the following FBSDE:

$$
\begin{cases}
X_t = x + \int_0^t \{A_s X_s - \bar{Y}_s + D_s \mathbb{E}[X_s] + \beta_s\} ds \\
\qquad\qquad + \int_0^t \{\sigma_s X_s + \alpha_s\} dW_s \\
\bar{Y}_t = (\sum_{i=1,m} K^i Q^i) X_T - \\
\int_t^T \{-A_s^\top \bar{Y}_s - (\sum_{i=1,m} K^i M_s^i) X_s - \sum_{j=1,m} (\sigma_s^j)^\top \bar{Z}_s^j \\
\qquad -\mathbb{E}[(\sum_{i=1,m} K^i \Gamma_s^i) X_s] - \mathbb{E}[D_s \bar{Y}_s]\} ds - \int_t^T \bar{Z}_s dW_s
\end{cases}
\tag{11}
$$

Remark: $\bar{Y} = \sum_{i=1,m} K^i p^i$.

By using Theorem of Section C, under (A), the solution $(X, \bar{Y}, \bar{Z})$ of this equation exists and is unique under smallness of D and $(\sum_{i=1,m} K^i \Gamma_s^i)$.

Next for $i = 1, m$, let us consider the following BSDE:

$$p_t^i = Q^i X_T + \int_t^T \{A_s^\top p_s^i + M_s^i X_s + \mathbb{E}[\Gamma_s^i X_s] + \mathbb{E}[D_s p_s^i] +$$

$$\sum_{j=1,m} (\sigma_s^j)^\top q_s^{i,j}\} ds - \int_t^T q_s^i dW_s, \; t \leq T.$$

The solution of this equation exists and is unique (BLP '09).

But multiplying p^i by K^i and summing wrt i we obtain that

$$(\sum_{i=1,m} K^i p^i, \sum_{i=1,m} K^i q^i)$$

is also solution of the BSDE part in (11).

Then by uniqueness of the solution of the BSDE we have

$$\bar{Y} = \sum_{i=1,m} K^i p^i \text{ and } \bar{Z} = \sum_{i=1,m} K^i q^i.$$

Next replacing $\bar{Y}$ by $\sum_{i=1,m} K^i p^i$ in the FBSDE (11) we obtain that $(X, (p^i, q^i)_{i=1,m})$ is a solution of the FBSDE of MKV type (2).

Theorem

The McKean-Vlasov NZSDG has an open-loop Nash equilibrium point under (A).

Thanks for your attention.

Joyeux Anniversaire Nicole. Merci pour ta générosité.