European options in a non-linear incomplete market with default

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Conference in honour of Nicole El Karoui

Market with imperfections

- Market with default .
  Ref : Jeanblanc, Blanchet-Scaillet, Crepey...
- The market is non-linear : the dynamics of the wealth process are non-linear.
  Ex : funding costs, repo rates, impact of a large investor on the default intensity...
- The market is incomplete

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- The market is incomplete
- Our goal : study of the superhedging price of a European option.

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- Let W be a one-dimensional Brownian motion.
- default time : τ random variable
- Let **N** be the default jump process :
  $\mathbf{N_t} := \mathbf{1}_{\tau \leq \mathbf{t}}$
- Let $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ be the filtration associated with W and N.
- **Hyp :** W is a $\mathbb{G}$-Brownian motion.'
- We have a $\mathbb{G}$-martingale representation theorem w.r.t. W and M (cf. Jeanblanc-Song (2015)).

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- $\mathbb{H}^2 := \{$ predictable processes Z s.t. $\mathbb{E}\left[\int_0^T Z_t^2 dt\right] < \infty\}$
- $\mathbb{H}^2_\lambda := \{$ predictable processes K s.t. $\mathbb{E}\left[\int_0^T K_t^2 \lambda_t dt\right] < \infty\}$

The market

One risky asset :

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t d\mathbf{M}_t) \text{ with } S_0 > 0.$$

- σ_t μ_t, and β_t are $\mathbb{G}$- predictable bounded.
- **Hyp :** $\sigma_t > 0$ and $\beta_\tau > -1$.
- To **simplify** the presentation, suppose $\sigma_t = \mathbf{1}$.

- investor with initial wealth x.
  $\mathbf{Z_t}$= amount invested in the risky asset at t (where $\mathbf{Z_t} \in \mathbb{H}^2$).
- Let $V_t^{x,Z}$ the value of the portfolio at time t.

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- Let $V_t^{x,Z}$ the value of the portfolio at time t.
- In the classical **linear** case :

$$dV_t = (r_t V_t + \theta_t Z_t)dt + Z_t(dW_t + \beta_t dM_t); \quad V_0 = x,$$

where r_t = risk-free interest rate, and $\theta_t := \mu_t - r_t$.

Here, for $x \in \mathbb{R}$ and a risky-asset strategy $\mathbf{Z} \in \mathbb{H}^2$, the wealth process $V_t^{x,Z}$ (or simply V_t) satisfies :

$$-dV_t = \mathbf{f}(t, V_t, Z_t)dt - Z_t dW_t - Z_t \beta_t dM_t; \quad V_0 = x.$$

where $\mathbf{f} : (t, \omega, y, z) \mapsto \mathbf{f}(t, \omega, y, z)$ is a **nonlinear** Lipschitz driver (non-convex).

Examples

recall the dynamics of the wealth $V^{x,Z}$:

$$-dV_t = \mathbf{f}(t, V_t, Z_t)dt - Z_t dW_t - Z_t \beta_t dM_t; \quad V_0 = x.$$

- Classical linear case : $f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t$, where $\theta_t = \mu_t - r_t$.
- borrowing rate $\mathbf{R} \neq$ lending rate $\mathbf{r}$: $f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t + (\mathbf{R_t} - \mathbf{r_t})(\mathbf{V_t} - \mathbf{Z_t})^-$.
- **a repo market** on which the risky asset is traded : $f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t - \mathbf{l_t Z_t^-} + \mathbf{b_t Z_t^+}$,
  $\mathbf{b_t}$ = borrowing repo rate,
  $\mathbf{l_t}$ = lending repo rate.
  (cf. Brigo, Rutkowski ...).
- **large seller** whose strategy impacts the default intensity (cf. Dum.-Grig.-Q.-Sul. (2018))

Pricing in a complete non-linear market

(Ref : El Karoui-P-Q 97) Brownian filtration : suppose $\mathcal{F} := \mathcal{F}^W$.

$$dS_t = S_t(\mu_t dt + dW_t)$$

Consider a European option with maturity T and payoff $\xi \in L^2(\mathcal{F}_T)$.
$\exists !$ (X, Z) in $\mathbb{H}^2 \times \mathbb{H}^2$ /

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t; \quad X_T = \xi.$$

$\rightarrow X = V^{X_0, Z}$

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$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t; \quad X_T = \xi.$$

$\rightarrow X = V^{X_0, Z} \rightarrow X_0 = X_0(T, \xi)$ is the hedging price(for the seller).

This leads to a f-nonlinear pricing system, introduced in El Karoui-Que. 96 : $(T, \xi) \mapsto X^f(T, \xi)$ satisfying the **monotonicity** property, **consistency** property /ξ, the **No-Arbitrage** property....
later called f-expectation (by S.Peng) and denoted by $\mathcal{E}^f$: $\forall \xi \in L^2(\mathcal{F}_T)$

$$\mathcal{E}^f_{s,T}(\xi) := X_s(T, \xi), s \in [0, T].$$

Here, our nonlinear market is **incomplete**.
Indeed, let $\xi \in L^2(\mathcal{G}_T)$. It might not be possible to find (x, Z) in $\mathbb{R} \times \mathbb{H}^2$ such that

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$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - \mathbf{K_t} dM_t; \quad Y_T = \xi.$$

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In general, $\mathbf{K} \neq Z\beta$.

Notation : if (Y, Z, K) is the solution of the $\mathbb{G}$-BSDE

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t; \quad Y_T = \xi,$$

we set $\mathcal{E}^f_{s,T}(\xi) := Y_s$ for all $s \in [0, T]$, called f-evaluation of ξ under P (with respect to $\mathcal{G}$).
Note that it might be a possible price but it does not allow the seller to be hedged.

Definition

seller's superhedging price at time 0 :

$$v_0 := \inf\{x \in \mathbb{R} : \exists Z \in \mathbb{H}^2 \text{ with } V_T^{x,Z} \geq \xi \ a.s.\}.$$

Dual representation formula for this price ?

The classical linear (incomplete) case

- *In this case*, $f(t,y,z) := -r_t y - \theta_t z$.
- **Definition** : Let $R \sim P$.
  R is called a martingale probability measure if
  $\forall\, x \in \mathbb{R}$, $\forall\, Z \in \mathbb{H}^2$, the process $(e^{-\int_0^t r_s ds} V_t^{x,Z})$ is an R-martingale
  (where $V^{x,Z}$ follows the linear dynamics with driver
  $f(t,y,z) := -r_t y - \theta_t z$).
- **Dual representation of the seller's superhedging price**
  (ref : EL Karoui-Qu.(91-95)) :

$$v_0 = \sup_{R \in \mathscr{P}} E_R(e^{-\int_0^T r_s ds} \xi),$$

where $\mathscr{P} := \{$ martingale probability measures$\}$.

Optional decomposition theorem in the linear case

Up to discounting, suppose that $r = 0$. We had first shown :

Theorem : (ref : EL Karoui-Qu.(91-95)), Föllmer...) :

Let (Y_t) *be an RCLL adapted process. If* (Y_t) *is an* R*-supermartingale* $\forall\, \mathbf{R} \in \mathscr{P}$*, then,* $\exists\, Z \in \mathbb{H}^2$*, and a nondecreasing optional RCLL process* $\mathbf{h}$*, with* $\mathbf{h}_0 = 0$ *such that*

$$Y_t = V_t^{Y_0, Z} - h_t \quad 0 \leq t \leq T.$$

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proof of the dual representation : let $X_S := ess\sup_{R \in \mathscr{P}} E_R(\xi | \mathcal{F}_S)$. By the above theorem, we show $X_t = V_t^{X_0,Z} - h_t$, $\forall t \in [0,T]$. Hence,

$$X_T = \xi = \mathbf{V_T^{X_0,Z}} - \mathbf{h_T} \Rightarrow V_T^{X_0,Z} \geq \xi \Rightarrow X_0 \geq v_0 \ldots X_0 = v_0. \textbf{ QED}$$

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Remark : $\forall R \in \mathscr{P}$, $E_R(\xi) = \mathbf{X_0} - \mathbf{E_R}(\mathbf{h_T})$.
Hence $\mathbf{inf_{R\in P} E_R(h_T) = 0}$

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Remark : $\forall R \in \mathscr{P}$, $E_R(\xi) = \mathbf{X_0} - \mathbf{E_R}(\mathbf{h_T})$.
Hence $\mathbf{inf_{R \in P} E_R(h_T) = 0}$ (h_T is the cumulated profit for the seller).

Similarly, $\forall S \leq T$, $X_S = ess\sup_{R \in \mathscr{P}} E_R(\xi \mid \mathcal{G}_S)$ is equal to the superhedging price at time S. As seen above,

$$X_t = V_t^{X_0, Z} - h_t, \forall t \leq T. \tag{2.1}$$

$\Rightarrow$ the profit process (for the seller) $(\mathbf{h_t})$ satisfies the **minimality** cn. :

$$ess \inf_{\mathbf{R} \in \mathscr{P}} E_R[h_T - h_S \,|\, \mathcal{G}_S] = 0 \quad \forall S \leq T. \tag{2.2}$$

Actually, we also have shown :

Theorem

Let X be any process with $X_T = \xi$ such that $\exists\ (Z, h)/$

$$X_t = V_t^{X_0, Z} - h_t, \forall t \leq T. \tag{2.3}$$

We have the equivalence property :
$X =$ superhedging price process $\Leftrightarrow$ the process $(\mathbf{h_t})$ satisfies (2.2).

- Question : what is the analogous of martingale probability measures in the case when f is non-linear ?
- First, we define the non-linear f-evaluation under Q.

Definition of W^Q and M^Q for $\mathbf{Q} \sim P$

From the $\mathbb{G}$-martingale representation theorem, its density process (ζ_t) satisfies

$$d\zeta_t = \zeta_{t^-}(\alpha_t dW_t + \nu_t dM_t); \zeta_0 = 1,$$

where (α_t) and (ν_t) are $\mathbb{G}$-predictable processes with $\nu_{\tau \wedge T} > -1$ a.s.
By Girsanov's theorem,

- $\mathbf{W^Q}_t := W_t - \int_0^t \alpha_s ds$ is a Q-Brownian motion, and
- $\mathbf{M^Q}_t := M_t - \int_0^t \nu_s \lambda_s ds$ is a Q-martingale.

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We have a Q-martingale representation for Q-martingales w.r.t. W^Q and M^Q.

f-evaluation under Q

Let $Q \sim P$.
We call f-evaluation under Q, denoted by $\mathcal{E}^f_Q$, the operator defined by :
for $\xi \in L^2_Q(\mathcal{G}_T)$,

$$\mathcal{E}^f_{\mathbf{Q},\mathbf{s},T}(\xi) := X_s, \qquad s \in [0, T]$$

where $(X, Z, K) \in \mathbb{H}^2_Q \times \mathbb{H}^2_Q \times \mathbb{H}^2_{Q,\lambda}$ satisfies the Q-**BSDE**

$$-dX_t = f(t, X_t, Z_t)dt - Z_t d\mathbf{W}^{\mathbf{Q}}_{\mathbf{t}} - K_t d\mathbf{M}^{\mathbf{Q}}_{\mathbf{t}}; \quad X_T = \xi.$$

Note that $\mathcal{E}^f_{\mathbf{P}} = \mathcal{E}^f$.

Non-linear expectation : $\mathcal{E}^f_{Q,t,T}(\xi) := X_t$, where (X, Z, K) satisfies :

$$-dX_t = f(t, X_t, Z_t)dt - Z_t d\mathbf{W}^{\mathbf{Q}}_{\mathbf{t}} - K_t d\mathbf{M}^{\mathbf{Q}}_{\mathbf{t}}; \quad X_T = \xi.$$

Definition

Let $Y \in S^2_Q$. The process (Y_t) is said to be a (strong) $\mathcal{E}^f_Q$-martingale (or (f, Q)-martingale), if $\forall \sigma, \tau \in \mathcal{T}$ with $\sigma \leq \tau$,

$$\mathcal{E}^f_{Q,\sigma,\tau}(Y_\tau) = Y_\sigma \quad \text{a.s.}.$$

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Question : what is the analogous of martingale probability measures in the non-linear case ?

Definition

A probability $Q \sim P$ is called an *f-martingale probability measure* if : $\forall\, x \in \mathbb{R}$ and $\forall\, Z \in \mathbb{H}^2_Q$, the wealth $V^{x,Z}$ is a (f, Q)-martingale.

We denote by $\mathcal{Q} := \{$ f-martingale probabilities $\}$

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Remarks :

- $P \in \mathcal{Q}$.
- $Q \in \mathcal{Q} \Leftrightarrow W + \int \beta_s dM_s$ is a Q-martingale.
- $\rightarrow$ The set $\mathcal{Q}$ does not depend on f.
- $\mathcal{Q}$ is equipotent to $\mathcal{P}$ (via a "translation" of θ)

Dual representation of the seller's price

Using the (f, Q)-martingale property of the wealths for $\mathbf{Q} \in \mathscr{Q}$, we easily show :

$$v_0 \geq \sup_{\mathbf{Q} \in \mathscr{Q}} \mathscr{E}^f_{\mathbf{Q},0,T}(\xi) = X(0),$$

where for each $S \in \mathcal{T}$,

$$X(S) := ess \sup_{\mathbf{Q} \in \mathscr{Q}} \mathscr{E}^f_{\mathbf{Q},S,T}(\xi).$$

Assumption : $E_{\mathbf{Q}}[ess\sup_{S \in \mathcal{T}} X(S)^2] < +\infty \ \forall \mathbf{Q} \in \mathscr{Q} \qquad (\Leftrightarrow v_0 < \infty).$

Theorem

$$v_0 = \sup_{\mathbf{Q} \in \mathscr{Q}} \mathscr{E}^f_{\mathbf{Q},0,T}(\xi).$$

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Theorem

$$v_0 = \sup_{\mathbf{Q} \in \mathcal{Q}} \mathcal{E}^f_{\mathbf{Q},0,T}(\xi).$$

Remark : The supremum is attained if and only if the option is replicable.

$\mathcal{E}^f$-optional decomposition

We first show :

Theorem :

Let $(Y_t) \in S^2_Q \; \forall \; Q \in \mathcal{Q}$.

If (Y_t) is a strong $\mathcal{E}^f_{\mathbf{Q}}$-supermartingale $\forall \; \mathbf{Q} \in \mathcal{Q}$,

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If (Y_t) is a strong $\mathscr{E}^f_{\mathbf{Q}}$-supermartingale $\forall \; \mathbf{Q} \in \mathscr{Q}$,

then, there exists $Z \in \mathbb{H}^2$, and a nondecreasing optional RCLL process $\mathbf{h}$, with $\mathbf{h}_0 = 0$ /

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)dt + \int_0^t Z_s(dW_s + \beta_s dM_s) - \mathbf{h}_t, \quad 0 \leq t \leq T.$$

This decomposition is unique.

Sketch of the proof of the dual representation :

$\exists\ (X_t) \in S^2 /$ for all S,

$$X_S = ess \sup_{\mathbf{Q} \in \mathscr{Q}} \mathscr{E}^f_{\mathbf{Q},S,T}(\xi) \quad a.s.$$

- It is an $\mathscr{E}^f_{\mathbf{Q}}$-supermartingale for each $\mathbf{Q} \in \mathscr{Q}$ (with $X(T) = \xi$).
- By the optional $\mathscr{E}^f$-decomposition theorem, $\exists\ Z, h... /$

$$X_t = X_0 - \int_0^t f(s, X_s, Z_s)dt + \int_0^t Z_s(dW_s + \beta_s dM_s) - \mathbf{h}_t, \quad 0 \leq t \leq T.$$

- By the comparison theorem for forward SDEs,

$$(\xi =)\ X_T \leq V_T^{X_0,Z}$$

Hence, $X_0 \geq v_0$. Since $X_0 \leq v_0$, we get $X_0 = v_0$. **QED**

Hyp. : $\sup_{\mathbf{Q}\in\mathcal{Q}} E_Q[\xi^2] < +\infty$.

We have seen that the superhedging price process X satisfies :

$X_t = X_0 - \int_0^t f(s, X_s, Z_s)dt + \int_0^t Z_s(dW_s + \beta_s dM_s) - \mathbf{h}_t = \mathbf{V_t^{X_0,Z,h}}$.

Proposition

The seller's cumulated profit $\mathbf{h_T}$ satisfies : $\inf_{\mathbf{Q}\in\mathcal{Q}} E_Q(\mathbf{h_T}) = 0$

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The seller's cumulated profit $\mathbf{h_T}$ satisfies : $\inf_{\mathbf{Q}\in\mathcal{Q}} E_Q(\mathbf{h_T}) = 0$

It can also be shown that

$$ess \inf_{\mathbf{Q}\in\mathcal{Q}} E_Q[h_T - h_S \,|\, \mathcal{G}_S] = 0 \quad \forall S \leq T. \tag{3.1}$$

Theorem

Let X be any process with $X_T = \xi$, such that $\exists\ (Z,h)/\ X_. = V_.^{X_0,Z,h}$.
Then,
$X_. =$ superhedging price process $\Leftrightarrow$ the process (h_t) satisfies (3.1).

Characterization of the superhedging price process via a constrained BSDE (with default)

Theorem : The seller's superhedging price process (X_t) is the (minimal) supersolution of the constrained BSDE with default : $\exists\,(Z,K) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ and **A predictable** nondecreasing such that

$$-dX_t = f(t,X_t,Z_t)dt - Z_t dW_t - K_t dM_t + d\mathbf{A}_t; \quad X_T = \xi\,;$$

$$\mathbf{A}_\cdot + \int_0^\cdot (K_s - \beta_s Z_s)\lambda_s ds \quad \text{is nondecreasing}$$

$$(K_t - \beta_t Z_t)\lambda_t \leq 0,\ dP \otimes dt - \mathrm{a.e.}\,;$$

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Remark : $\mathbf{h}_t = \mathbf{A}_t - \int_0^t (K_s - \beta_s Z_s) dM_s$.

Grigorova M., Quenez M.-C. and A. Sulem : Non-linear pricing of European options in an incomplete market with default, (2018), preprint.

Dumitrescu D., Grigorova M., Quenez M.-C. and A. Sulem : BSDE with default jump, (2019), Computation and Combinatorics in Dynamics, Stochastics and Control - The Abel Symposium 2016, 13, Springer.