European options
in a non-linear incomplete market with default

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Conference in honour of Nicole El Karoui
Market with imperfections

- Market with **default**.
  Ref: Jeanblanc, Blanchet-Scaillet, Crepey...
- The market is **non-linear**: the dynamics of the wealth process are non-linear.
  Ex: funding costs, repo rates, impact of a large investor on the default intensity...
- The market is **incomplete**
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- The market is **incomplete**
- Our goal: study of the **superhedging price** of a European option.
The model

- Let $(\Omega, \mathcal{G}, \mathcal{P})$ be a complete probability space.
- Let $W$ be a one-dimensional Brownian motion.
- Default time: $\tau$ random variable
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- Let \((\Omega, \mathcal{G}, \mathcal{P})\) be a complete probability space.
- Let \(W\) be a one-dimensional Brownian motion.
- **default time**: \(\tau\) random variable
- Let \(\mathbf{N}\) be the **default jump process**:
  \[ \mathbf{N}_t := 1_{\tau \leq t} \]
- Let \(\mathcal{G} = \{\mathcal{G}_t, t \geq 0\}\) be the filtration associated with \(W\) and \(\mathbf{N}\).
- **Hyp**: \(W\) is a \(\mathcal{G}\)-Brownian motion.
- We have a \(\mathcal{G}\)-martingale representation theorem w.r.t. \(W\) and \(M\) (cf. Jeanblanc-Song (2015)).
**Hyp** : the $\mathcal{G}$-predictable compensator of $N_t$ is $\int_0^t \lambda_s \, ds$. $(\lambda_s)$ is called the intensity process, and is supposed to be bounded. It vanishes after $\tau$. 

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**Hyp**: the $\mathcal{G}$-predictable compensator of $N_t$ is: $\int_0^t \lambda_s ds$. ($\lambda_s$) is called the intensity process, and is supposed to be bounded. It vanishes after $\tau$.

The compensated martingale of $(N_t)$ is thus given by

$$M_t := N_t - \int_0^t \lambda_s ds$$
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$H^2 := \{ \text{predictable processes } Z \text{ s.t. } \mathbb{E} \left[ \int_0^T Z^2_t \, dt \right] < \infty \}$

$H^2_{\lambda} := \{ \text{predictable processes } K \text{ s.t. } \mathbb{E} \left[ \int_0^T K^2_t \lambda_t \, dt \right] < \infty \}$
Introduction

The market

One risky asset:

\[ dS_t = S_t \left( \mu_t dt + \sigma_t dW_t + \beta_t dM_t \right) \] with \( S_0 > 0 \).

- \( \sigma_t \mu_t \), and \( \beta_t \) are \( \mathbb{G} \)-predictable bounded.
- Hyp: \( \sigma_t > 0 \) and \( \beta_t > -1 \).
- To simplify the presentation, suppose \( \sigma_t = 1 \).

- investor with initial wealth \( x \).
- \( Z_t \) = amount invested in the risky asset at \( t \) (where \( Z_t \in \mathbb{H}^2 \)).
- Let \( V_{t}^{x,Z} \) the value of the portfolio at time \( t \).
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- In the classical linear case:

\[ dV_t = (r_t V_t + \theta_t Z_t) dt + Z_t (dW_t + \beta_t dM_t); \quad V_0 = x, \]

where \( r_t = \) risk-free interest rate, and \( \theta_t := \mu_t - r_t \).
Here, for $x \in \mathbb{R}$ and a risky-asset strategy $Z \in \mathbb{H}^2$, the wealth process $V_{t}^x, Z$ (or simply $V_t$) satisfies:

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t \beta_t dM_t; \quad V_0 = x.$$ 

where $f : (t, \omega, y, z) \mapsto f(t, \omega, y, z)$ is a nonlinear Lipschitz driver (non-convex).
Examples

recall the dynamics of the wealth $V^{x,Z}$:

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t \beta_t dM_t; \quad V_0 = x.$$ 

- **Classical linear case**: $f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t$, where $\theta_t = \mu_t - r_t$.

- **borrowing rate** $R \neq$ lending rate $r$:
  
  $$f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t + (R_t - r_t)(V_t - Z_t)^-. $$

- **a repo market** on which the risky asset is traded:
  
  $$f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t - I_t Z^- + b_t Z^+, $$

  $b_t$ = borrowing repo rate,
  $l_t$ = lending repo rate.

  (cf. Brigo, Rutkowski ...).

- **large seller** whose strategy impacts the default intensity (cf. Dum.-Grig.-Q.-Sul. (2018))
Pricing in a complete non-linear market
(Ref: El Karoui-P-Q 97) Brownian filtration: suppose $\mathcal{F} := \mathcal{F}^W$.

$$dS_t = S_t(\mu_t dt + dW_t)$$

Consider a European option with maturity $T$ and payoff $\xi \in L^2(\mathcal{F}_T)$.

$\exists! (X, Z)$ in $\mathbb{H}^2 \times \mathbb{H}^2$

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t; \quad X_T = \xi.$$

$\rightarrow X = V^{X_0, Z}$
Pricing in a complete non-linear market

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$\exists! (X, Z)$ in $H^2 \times H^2$ / 

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t; \quad X_T = \xi.$$ 

$\rightarrow X = V^{X_0, Z} \rightarrow X_0 = X_0(T, \xi)$ is the hedging price (for the seller).

This leads to a $f$-nonlinear pricing system, introduced in El Karoui-Que. 96 : 

$(T, \xi) \mapsto X^f(T, \xi)$ satisfying the monotonicity property, consistency property /$\xi$, the No-Arbitrage property....

later called $f$-expectation (by S.Peng) and denoted by $\mathcal{E}^f : \forall \xi \in L^2(\mathcal{F}_T)$

$$\mathcal{E}^f_{s, T}(\xi) := X_s(T, \xi), s \in [0, T].$$
Here, our nonlinear market is **incomplete**.
Indeed, let $\xi \in L^2(G_T)$. It might not be possible to find $(x, Z)$ in $\mathbb{R} \times \mathbb{H}^2$ such that

$$V_T^{x,Z} = \xi.$$
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In other words, there does not necessarily exist $(V, Z) \in \mathbb{H}^2 \times \mathbb{H}^2/\lambda$

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t \beta_t dM_t; \quad V_T = \xi,$$

however, by the $G$-martingale representation w.r.t. $W, M$, there exists $(Y, Z, K)$ in $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$ solution of the BSDE with default (cf. G-Q-S 2017 for details)

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t; \quad Y_T = \xi.$$
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In general, $K \neq Z\beta$. 

Notation : if \((Y, Z, K)\) is the solution of the \(\mathbb{G}\)-BSDE

\[-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t; \quad Y_T = \xi,\]

we set \(\mathbb{G}_s^f(\xi) := Y_s\) for all \(s \in [0, T]\), called \(f\)-evaluation of \(\xi\) under \(P\) (with respect to \(\mathcal{G}\)).

Note that it might be a possible price but it does not allow the seller to be hedged.

**Definition**

seller’s superhedging price at time 0 : 

\[v_0 := \inf\{x \in \mathbb{R} : \exists Z \in \mathbb{H}^2 \text{ with } V_T^x,Z \geq \xi \text{ a.s.}\}.

Dual representation formula for this price ?
The classical linear (incomplete) case

- **In this case**, \( f(t, y, z) := -r_t y - \theta_t z \).

- **Definition**: Let \( R \sim P \).
  
  \( R \) is called a **martingale probability measure** if 
  
  \( \forall x \in \mathbb{R}, \forall Z \in \mathcal{H}^2, \) the process \( (e^{-\int_0^t r_s ds} V^{x,Z}_t) \) is an \( R \)-martingale 
  
  (where \( V^{x,Z} \) follows the linear dynamics with driver 
  
  \( f(t, y, z) := -r_t y - \theta_t z \)).

- **Dual representation of the seller’s superhedging price**
  
  (ref: EL Karoui-Qu.(91-95)):

  \[
  \nu_0 = \sup_{R \in \mathcal{P}} E_R(e^{-\int_0^T r_s ds} \xi),
  \]

  where \( \mathcal{P} := \{ \text{martingale probability measures} \} \).
Optional decomposition theorem in the linear case

Up to discounting, suppose that $r = 0$. We had first shown:

**Theorem**: (ref: EL Karoui-Qu.(91-95), Föllmer...):

Let $(Y_t)$ be an RCLL adapted process. If $(Y_t)$ is an $R$-supermartingale \( \forall R \in \mathcal{P} \), then, \( \exists Z \in \mathbb{H}^2 \), and a nondecreasing optional RCLL process $h$, with $h_0 = 0$ such that

\[
Y_t = V_t^{Y_0,Z} - h_t \quad 0 \leq t \leq T.
\]

**proof of the dual representation**:

Let $X_S := \text{ess sup}_{R \in \mathcal{P}} E_R(\xi | F_S)$.

By the above theorem, we show $X_t = V^{X_0,Z}_t$, $Z_t - h_t$, $\forall t \in [0,T]$. Hence, $X_T = \xi = V^{X_0,Z}_0$, $Z_T - h_T \Rightarrow V^{X_0,Z}_0$, $Z_T \geq \xi \Rightarrow X_0 \geq v_0$... $X_0 = v_0$.

**QED**

**Remark**: \( \forall R \in \mathcal{P} \), $E_R(\xi) = X_0 - E_R(h_T)$.

Hence $\inf_{R \in \mathcal{P}} E_R(h_T) = 0$ (\( h_T \) is the cumulated profit for the seller).
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$$X_T = \xi = V_T^{X_0,Z} - h_T \implies V_T^{X_0,Z} \geq \xi \implies X_0 \geq v_0 \ldots X_0 = v_0. \text{ QED}$$
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**Remark**: \( \forall R \in \mathcal{P}, E_R(\xi) = X_0 - E_R(h_T) \).

Hence \( \inf_{R \in \mathcal{P}} E_R(h_T) = 0 \) (\( h_T \) is the cumulated profit for the seller).
Similarly, \( \forall S \leq T, \ X_S = \text{ess sup}_{R \in \mathcal{P}} \ E_R(\xi \mid G_S) \) is equal to the superhedging price at time \( S \). As seen above,

\[
X_t = V_t^{X_0, Z} - h_t, \forall t \leq T. \tag{2.1}
\]

\( \Rightarrow \) the profit process (for the seller) \((h_t)\) satisfies the \textit{minimality} con.:

\[
\text{ess inf}_{R \in \mathcal{P}} \ E_R[h_T - h_S \mid G_S] = 0 \quad \forall S \leq T. \tag{2.2}
\]

Actually, we also have shown:

**Theorem**

Let \( X \) be any process with \( X_T = \xi \) such that \( \exists \ (Z, h) \):

\[
X_t = V_t^{X_0, Z} - h_t, \forall t \leq T. \tag{2.3}
\]

We have the equivalence property:

\( X = \text{superhedging price process} \iff \text{the process} \ (h_t) \ \text{satisfies} \ (2.2). \)
Question: what is the analogous of martingale probability measures in the case when $f$ is non-linear?

First, we define the non-linear $f$-evaluation under $Q$. 
Definition of $W^Q$ and $M^Q$ for $Q \sim P$

From the $\mathbb{G}$-martingale representation theorem, its density process $(\zeta_t)$ satisfies

$$d\zeta_t = \zeta_t - (\alpha_t dW_t + \nu_t dM_t); \zeta_0 = 1,$$

where $(\alpha_t)$ and $(\nu_t)$ are $\mathbb{G}$-predictable processes with $\nu_{\tau\wedge T} > -1$ a.s.

By Girsanov’s theorem,

- $W^Q_t := W_t - \int_0^t \alpha_s ds$ is a $Q$-Brownian motion, and
- $M^Q_t := M_t - \int_0^t \nu_s \lambda_s ds$ is a $Q$-martingale.
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We have a $Q$-martingale representation for $Q$-martingales w.r.t. $W^Q$ and $M^Q$. 
The seller’s price: dual representation

**f**-evaluation under \( Q \)

Let \( Q \sim P \).

We call \( f \)-evaluation under \( Q \), denoted by \( \mathcal{E}_Q^f \), the operator defined by:

for \( \xi \in L^2_Q(G_T) \),

\[
\mathcal{E}_Q^f(s, T)(\xi) := X_s, \quad s \in [0, T]
\]

where \( (X, Z, K) \in \mathbb{H}^2_Q \times \mathbb{H}^2_Q \times \mathbb{H}^2_{Q, \lambda} \) satisfies the \( Q \)-BSDE

\[
-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t^Q - K_t dM_t^Q; \quad X_T = \xi.
\]

Note that \( \mathcal{E}_P^f = \mathcal{E}_Q^f \).
Non-linear expectation: $\mathcal{E}^f_{Q,t,T}(\xi) := X_t$, where $(X, Z, K)$ satisfies:

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW^Q_t - K_t dM^Q_t; \quad X_T = \xi.$$ 

**Definition**

Let $Y \in S^2_Q$. The process $(Y_t)$ is said to be a (strong) $\mathcal{E}^f_Q$-martingale (or $(f, Q)$-martingale), if $\forall \sigma, \tau \in T$ with $\sigma \leq \tau$,

$$\mathcal{E}^f_{Q,\sigma,\tau}(Y_\tau) = Y_\sigma \quad \text{a.s..}$$
Non-linear expectation: \( \mathcal{E}_{Q,t,T}^f(\xi) := X_t \), where \((X, Z, K)\) satisfies:

\[-dX_t = f(t, X_t, Z_t) \, dt - Z_t \, dW_t^Q - K_t \, dM_t^Q; \quad X_T = \xi.\]

**Definition**

Let \( Y \in S_Q^2 \). The process \((Y_t)\) is said to be a (strong) \( \mathcal{E}_Q^f \)-martingale (or \((f, Q)\)-martingale), if \( \forall \sigma, \tau \in \mathcal{T} \) with \( \sigma \leq \tau \),

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**Question**: what is the analogous of martingale probability measures in the non-linear case?
**Definition**

A probability $Q \sim P$ is called an $f$-martingale probability measure if:

$\forall x \in \mathbb{R}$ and $\forall Z \in \mathbb{H}_Q^2$, the wealth $V^{x,Z}$ is a $(f, Q)$-martingale.

We denote by $\mathcal{Q} := \{ f$-martingale probabilities $\}$.
Definition
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Remarks:
- $P \in \mathcal{Q}$.
- $Q \in \mathcal{Q} \iff W + \int \beta_s dM_s$ is a $Q$-martingale.
- $\rightarrow$ The set $\mathcal{Q}$ does not depend on $f$.
- $\mathcal{Q}$ is equipotent to $\mathcal{P}$ (via a "translation" of $\theta$)
The seller’s price: dual representation

Dual representation of the seller’s price

Using the \((f, Q)\)-martingale property of the wealths for \(Q \in \mathcal{Q}\), we easily show:

\[
v_0 \geq \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q^{f} \xi = X(0),
\]

where for each \(S \in \mathcal{T}\),

\[
X(S) := \text{ess sup}_{Q \in \mathcal{Q}} \mathbb{E}_Q^{f} \xi_S = \mathbb{E}_Q^{f} \xi_T(X(0))
\]

Assumption: \(E_Q[\text{ess sup}_{S \in \mathcal{T}} X(S)^2] < +\infty \ \forall Q \in \mathcal{Q} \quad (\iff v_0 < \infty)\).

Theorem

\[
v_0 = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q^{f} \xi = X(0).
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Theorem

\[
v_0 = \sup_{Q \in \mathcal{Q}} \mathbb{E}_{Q,0,T}^f(\xi).
\]

Remark: The supremum is attained if and only if the option is replicable.
\( \mathcal{E}^f \)-optional decomposition

We first show:

**Theorem:**
Let \((Y_t) \in S_Q^2 \forall Q \in \mathcal{Q}\).
If \((Y_t)\) is a strong \(\mathcal{E}_Q^f\)-supermartingale \(\forall Q \in \mathcal{Q}\),

This decomposition is unique.
We first show:

**Theorem:**

Let \(( Y_t ) \in S^2_Q \forall Q \in \mathcal{Q}.

If \(( Y_t )\) is a strong \(\mathcal{E}^f_Q\)-supermartingale \(\forall Q \in \mathcal{Q}\), then, there exists \(Z \in \mathcal{H}^2\), and a nondecreasing optional RCLL process \(h\), with \(h_0 = 0 / \)

\[
Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s) \, ds + \int_0^t Z_s (dW_s + \beta_s dM_s) - h_t, \quad 0 \leq t \leq T.
\]

This decomposition is unique.
Sketch of the proof of the dual representation:
\[ \exists (X_t) \in S^2/ \text{for all } S, \]

\[ X_S = \text{ess sup}_{Q \in \mathcal{Q}} \mathcal{E}_Q^{f} S, T(\xi) \quad \text{a.s.} \]

- It is an \( \mathcal{E}_Q^{f} \)-supermartingale for each \( Q \in \mathcal{Q} \) (with \( X(T) = \xi \)).
- By the optional \( \mathcal{E}^{f} \)-decomposition theorem, \( \exists Z, h... / \)

\[ X_t = X_0 - \int_0^t f(s, X_s, Z_s) dt + \int_0^t Z_s (dW_s + \beta_s dM_s) - h_t, \quad 0 \leq t \leq T. \]

- By the comparison theorem for forward SDEs,

\[ (\xi =) X_T \leq V^{X_0, Z}_T \]

Hence, \( X_0 \geq \nu_0 \). Since \( X_0 \leq \nu_0 \), we get \( X_0 = \nu_0 \). QED
Hyp.: $\sup_{Q \in \mathcal{Q}} E_Q[\xi^2] < +\infty$.

We have seen that the superhedging price process $X$ satisfies:

$$X_t = X_0 - \int_0^t f(s, X_s, Z_s) \, ds + \int_0^t Z_s (dW_s + \beta_s \, dM_s) - h_t = V_t^{X_0, Z, h}.$$ 

**Proposition**

The seller’s cumulated profit $h_T$ satisfies:

$$\inf_{Q \in \mathcal{Q}} E_Q(h_T) = 0.$$
**Hyp.** : \( \sup_{Q \in \mathcal{Q}} E_Q[\xi^2] < +\infty \).

We have seen that the superhedging price process \( X \) satisfies:

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**Proposition**

The seller’s cumulated profit \( h_T \) satisfies:

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\]

It can also be shown that

\[
\text{ess inf}_{Q \in \mathcal{Q}} E_Q[h_T - h_S \mid G_S] = 0 \quad \forall S \leq T.
\]  \hspace{1cm} (3.1)

**Theorem**

Let \( X \) be any process with \( X_T = \xi \), such that \( \exists (Z, h) \mid X = V^{X_0, Z, h} \).

Then, \( X \) = superhedging price process if and only if the process \( (h_t) \) satisfies (3.1).
Characterization of the superhedging price process via a constrained BSDE (with default)

**Theorem** : The seller’s superhedging price process \((X_t)\) is the (minimal) supersolution of the constrained BSDE with default :

\[ \exists (Z, K) \in H^2 \times H^2_{\lambda} \text{ and A predictable nondecreasing such that} \]

\[-dX_t = f(t, X_t, Z_t) dt - Z_t dW_t - K_t dM_t + dA_t; \quad X_T = \xi; \]

\[ A. + \int_0^t (K_s - \beta_s Z_s) \lambda_s ds \text{ is nondecreasing} \]

\[(K_t - \beta_t Z_t) \lambda_t \leq 0, \; dP \otimes dt - \text{a.e.}; \]
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\[
(K_t - \beta_t Z_t) \lambda_t \leq 0, \quad dP \otimes dt \text{ – a.e.};
\]

**Remark**: \(h_t = A_t - \int_0^t (K_s - \beta_s Z_s) dM_s.\)