CONSERVATIVE DIFFUSION AS ENTROPIC FLOW OF STEEPEST DESCENT

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OUTLINE

We provide a probabilistic interpretation, based on stochastic calculus, for the variational characterization of diffusion as **entropic gradient flux.** This was first established, through a discretization scheme, in the seminal paper by Jordan, Kinderlehrer and Otto (1998). [1]

It was shown by those authors that, for diffusions of the Langevin-Smoluchowski type

$$\mathrm{d}X(t) = -\nabla\Psi\big(X(t)\big)\,\mathrm{d}t + \mathrm{d}W(t)\,,$$

the associated Fokker-Planck probability density flow *minimizes the rate of relative entropy dissipation* – as measured by the distance traveled in the ambient space of probability measures with finite second moments, as measured by the quadratic Wasserstein metric on that space.

[1] It was then extended to more general settings via a set of tools that became known colloquially as "Otto Calculus".

We obtain novel, stochastic-process versions of these features, valid along *almost every trajectory of the diffusive motion* in both the forward and, most transparently, the **backward** (as in Fontbona-Jourdain 2016), directions of time, using a very direct perturbation analysis.

By averaging our trajectorial results with respect to the underlying measure on path space, we establish the minimum rate of entropy dissipation along the Fokker-Planck flow – and measure precisely the deviation from this minimum that corresponds to any given perturbation. [2]

As a bonus of this perturbation analysis, the so-called *HWI inequality* relating relative entropy (H), Wasserstein distance (W) and relative Fisher information (I), literally falls in our lap. [3]

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[2] Our approach can be described pithily as "carrying out and extending Otto calculus by means of Itô's calculus".

[3] And with it the Talagrand, log-Sobolev and Poincaré inequalities.

THE SETTING

Start with a "smooth potential well" $\Psi : \mathbb{R}^n \to [0, \infty)$.
Think quadratic,

$$\Psi(x) = |x|^2 \,;$$

but our conditions will allow "double well" potentials of the sort $\Psi(x) = \left(x^2 - a^2\right)^2$, or even
$\Psi(x) \equiv 0$.

We place Brownian particles in such a potential well. They diffuse, but also "slide along the edges of the well", according to the Langevin-Smoluchowski equation

$$\mathrm{d}X(t) = -\nabla\Psi\big(X(t)\big)\,\mathrm{d}t + \mathrm{d}W(t)\,.$$

Because of the sliding towards the bottom of the well, this motion is “conservative”: has an invariant distribution Q on $\mathcal{B}(\mathbb{R}^n)$ (the so-called “Gibbs measure”), with density

$$q(x) := e^{-2\Psi(x)}$$

relative to Lebesgue measure.

No need to assume that this Q is finite (so it can be normalized to a probability):
It is just a $\sigma-$finite measure.

We posit now an initial distribution $P(0)$ of particles with density $p_0(\cdot)$ that admits finite second moment: $\int_{\mathbb{R}^n} |x|^2 p_0(x) \mathrm{d}x < \infty$.

. Under the coercivity condition

$$\big\langle x, \nabla \Psi(x) \big\rangle_{\mathbb{R}^n} \geq -c\,|x|^2\,, \qquad \forall \;\; |x| > R\,,$$

for suitable positive real constants c, R, such finiteness propagates:

. With $P(t)$ the distribution of particles at time $t \in [0, \infty)$, the corresponding density $p(t, \cdot)$ also admits a finite second moment: $\int_{\mathbb{R}^n} |x|^2 p(t, x) \mathrm{d}x < \infty$.

This probability density function satisfies the Fokker-Planck (or **forward** Kolmorogov) equation

$$\partial_t p(t, x) \;=\; \frac{1}{2}\,\Delta p(t, x) + \operatorname{div}\big(\nabla \Psi(x)\, p(t, x)\big).$$

WASSERSTEIN DISTANCE

The coercivity condition ensures that the resulting flow of probability measures

$$\big(P(t)\big)_{0 \le t < \infty}, \qquad P(t, A) = \int_A p(t, x)\, \mathrm{d}x$$

is a curve in

$$\mathcal{P}_2(\mathbb{R}^n),$$

the so-called quadratic Wasserstein space of probability measures with finite second moment and endowed with the familiar distance

$$W(\mu, \nu) := \left(\inf_{\substack{Y \sim \mu \\ Z \sim \nu}} E\left|Y - Z\right|^2 \right)^{1/2}.$$

This $\mathcal{P}_2(\mathbb{R}^n)$ is the ambient space, where the configuration of our system will live.

RELATIVE ENTROPY

For every probability measure $P \in \mathcal{P}_2(\mathbb{R}^n)$ in this quadratic Wasserstein space, it is possible to define the relative entropy with respect to the invariant measure Q, as

$$H(P|Q) := \int_{\mathbb{R}^n} \log\left(\frac{\mathrm{d}P}{\mathrm{d}Q}\right) \mathrm{d}P \quad \in \quad (-\infty, \infty]$$

if $P \ll Q$; and as $H(P|Q) := \infty$ otherwise. We are relying here on the construction of Ch. Léonard.

IF this invariant measure Q is a probability, then the relative entropy right above is a non-negative quantity.

ENERGY AND ENTROPY

For a probability density function $\rho : \mathbb{R}^n \to [0, \infty)$, let us introduce the potential energy and the Gibbs-Boltzmann entropy

$$E(\rho) := \int_{\mathbb{R}^n} \Psi(x)\, \rho(x)\, dx\,, \qquad S(\rho) := \int_{\mathbb{R}^n} \rho(x)\, \log \rho(x)\, dx\,,$$

respectively, as well as the "free energy" (sum of potential and internal energies)

$$F(\rho) := E(\rho) + \frac{1}{2}\, S(\rho)\,.$$

We shall assume that our initial configuration of particles has finite free energy:

$$F\big(p_0(\cdot)\big) < \infty\,.$$

. Then this free energy decreases along the flow of probability density functions $\big(p_t(\cdot)\big)_{0 \le t < \infty}$, or equivalently along the flow of the corresponding probability measures

$$\big(P(t)\big)_{0 \le t < \infty} \subset \mathcal{P}_2(\mathbb{R}^n)\,.$$

. And is a constant multiple of the relative entropy with respect to the invariant distribution: to wit, the function

$$t \longmapsto 2\,F\big(p_t(\cdot)\big) \;=\; H\big(P(t)\big|Q\big) \in \mathbb{R} \quad \text{decreases}$$

in accordance with the second law of thermodynamics (more about this decrease, and its temporal and ambient rates, in a moment).

PROBABILISTIC SETTING AND NOTATION

Let us denote by $\mathbb{P}$ the probability measure induced by the diffusion $X(\cdot)$ governed by the equation

$$\mathrm{d}X(t) = -\nabla\Psi\big(X(t)\big)\,\mathrm{d}t + \mathrm{d}W(t)$$

on the space $C([0,\infty);\mathbb{R}^n)$ of continuous functions.

We introduce the likelihood ratio **function**

$$\ell(t,x) := \frac{p(t,x)}{q(x)} = p(t,x)\,e^{2\,\Psi(x)}\,, \qquad x \in \mathbb{R}^n$$

and the likelihood ratio **process**

$$\ell\big(t,X(t)\big) = \frac{\mathrm{d}P(t)}{\mathrm{d}Q}\big(X(t)\big)\,, \qquad 0 \le t < \infty\,.$$

Then, we have the representation of the relative entropy

$$H\big(P(t)\big|Q\big) \;=\; \mathbb{E}^{\mathbb{P}}\big[\,\log \ell\big(t, X(t)\big)\,\big]$$

$$= \int_{\mathbb{R}^n} \Big(\log p(t,x) + 2\Psi(x)\big)\Big) p(t,x)\,\mathrm{d}x\,,$$

and the definition of the Fisher information

$$I\big(P(t)\big|Q\big) \;:=\; \mathbb{E}^{\mathbb{P}}\Big[\,\big|\nabla \log \ell\big(t, X(t)\big)\big|^2\,\Big]$$

$$= \int_{\mathbb{R}^n} \Big|\nabla\big(\log p(t,x) + 2\Psi(x)\big)\Big|^2 p(t,x)\,\mathrm{d}x\,.$$

BASIC IDENTITIES

We have the (HI, or **de Bruijn**), (WI) and (HWI) identities

$$\lim_{t \downarrow t_0} \frac{H\big(P(t)\big|Q\big) - H\big(P(t_0)\big|Q\big)}{t - t_0} = -\frac{1}{2}\, I\big(P(t)\big|Q\big)$$

$$\lim_{t \downarrow t_0} \frac{W\big(P(t), P(t_0)\big)}{t - t_0} = \frac{1}{2}\sqrt{I\big(P(t)\big|Q\big)}$$

$$\lim_{t \downarrow t_0} \frac{H\big(P(t)\big|Q\big) - H\big(P(t_0)\big|Q\big)}{W\big(P(t), P(t_0)\big)} = -\sqrt{I\big(P(t)\big|Q\big)}\,.$$

. The (negative) quantity in this last identity, measures the rate of decrease ("descent") of relative entropy along the curve $\big(P(t)\big)_{0 \le t < \infty}$ in terms of the distance traveled in the ambient Wasserstein space $\mathcal{P}_2(\mathbb{R}^n)$.

. *We shall see also that this descent is the "steepest possible", in a sense we'll make very precise presently.*

PERTURBATION ANALYSIS

Keep the same dynamics for the diffusive particles, on $[0, t_0]$; but from that time onward, perturb the drift by the gradient

$$\beta = \nabla B$$

of a smooth, compactly supported potential:

$$\mathrm{d}X(t) = -\big[\nabla\Psi\big(X(t)\big) + \nabla B\big(X(t)\big)\big]\mathrm{d}t + \mathrm{d}W^{\beta}(t)\,, \quad t > t_0\,.$$

Denote by $\mathbb{P}^{\beta}$ the probability measure induced in this manner on the space $C([0,\infty);\mathbb{R}^n)$ of continuous functions, under which W^{β} is Brownian motion.

And denote by $P^{\beta}(t)$ the $\mathbb{P}^{\beta}-$distribution of the variable $X(t)$; clearly, $P^{\beta}(t_0) \equiv P(t_0)$.

We introduce the random vectors

$$\mathfrak{a} := \nabla \log \ell\big(t_0, X(t_0)\big)\,, \qquad \mathfrak{b} := \beta\big(X(t_0)\big)\,.$$

in $L^2(\mathbb{P})$, and denote by

$$\big\langle\big\langle\, \mathfrak{a}, \mathfrak{b}\,\big\rangle\big\rangle_{L^2(\mathbb{P})}\,, \qquad \big\|\,\mathfrak{a}\,\big\|_{L^2(\mathbb{P})}$$

inner product and norm, respectively, in this space $L^2(\mathbb{P})$.

BASIC IDENTITIES: THE PERTURBED CASE

Then we have the **"perturbed"** versions of the (HI), (WI) and (HWI) identities

$$\lim_{t \downarrow t_0} \frac{H\big(P^\beta(t)\big|Q\big) - H\big(P^\beta(t_0)\big|Q\big)}{t - t_0} = -\frac{1}{2} \big\langle\big\langle\, \mathfrak{a}, \mathfrak{a} + 2\mathfrak{b} \,\big\rangle\big\rangle_{L^2(\mathbb{P})}$$

$$\lim_{t \downarrow t_0} \frac{W\big(P^\beta(t), P^\beta(t_0)\big)}{t - t_0} = \frac{1}{2} \big\|\, \mathfrak{a} + 2\mathfrak{b} \,\big\|_{L^2(\mathbb{P})}$$

$$\lim_{t \downarrow t_0} \frac{H\big(P^\beta(t)\big|Q\big) - H\big(P^\beta(t_0)\big|Q\big)}{W\big(P^\beta(t), P^\beta(t_0)\big)} = -\left\langle\!\!\left\langle \mathfrak{a}\,, \frac{\mathfrak{a} + 2\mathfrak{b}}{\big|\mathfrak{a} + 2\mathfrak{b}\big|} \right\rangle\!\!\right\rangle_{L^2(\mathbb{P})} .$$

We recall here from the previous slide the random vectors

$$\mathfrak{a} := \nabla \log \ell\big(t_0, X(t_0)\big)\,, \qquad \mathfrak{b} := \beta\big(X(t_0)\big).$$

Now compare this last "perturbed" slope

$$S^\beta(t_0) := \lim_{t \downarrow t_0} \frac{H\big(P^\beta(t)\big|Q\big) - H\big(P^\beta(t_0)\big|Q\big)}{W\big(P^\beta(t), P^\beta(t_0)\big)}$$

$$= - \left\langle\!\!\left\langle \mathfrak{a}\,, \frac{\mathfrak{a} + 2\mathfrak{b}}{\big|\mathfrak{a} + 2\mathfrak{b}\big|} \right\rangle\!\!\right\rangle_{L^2(\mathbb{P})}$$

with the "unperturbed" slope

$$S(t_0) := \lim_{t \downarrow t_0} \frac{H\big(P(t)\big|Q\big) - H\big(P(t_0)\big|Q\big)}{W\big(P(t), P(t_0)\big)} = - \big\| \, \mathfrak{a} \, \big\|_{L^2(\mathbb{P})}$$

from a few slides upstream.

Their difference in non-negative

$$S^\beta(t_0) - S(t_0) = \| \mathfrak{a} \|_{L^2(\mathbb{P})} - \left\langle\!\!\left\langle \mathfrak{a}, \frac{\mathfrak{a}+2\mathfrak{b}}{|\mathfrak{a}+2\mathfrak{b}|} \right\rangle\!\!\right\rangle_{L^2(\mathbb{P})} \geq 0,$$

in fact strictly positive unless the random vectors $\mathfrak{a} = \nabla \log \ell\big(t_0, X(t_0)\big)$, $\mathfrak{b} = \beta\big(X(t_0)\big)$ are collinear.

. The **(negative)** slope along the original, unperturbed Fokker-Planck flow $\big(P(t)\big)_{t \geq 0}$, namely

$$S(t_0) := \lim_{t \downarrow t_0} \frac{H\big(P(t)\big|Q\big) - H\big(P(t_0)\big|Q\big)}{W\big(P(t), P(t_0)\big)} = -\| \mathfrak{a} \|_{L^2(\mathbb{P})},$$

is the slope of **steepest descent** among all such perturbations. This is the "entropic gradient flux" property of the title.

THE JOYS OF PROBABILITY

How do we get to all this? The short answer is,

"Well, we let the trajectories do all the work for us;
then we just go in, and take expectations."

Or, as Nicole El Karoui used to say to me, when we were working together:

"Elles sont jolies, les Probas!"

TIME REVERSAL

But here with a twist: **It pays to look at trajectories in the reverse direction of time.**

A very strong hint, as to why such an approach might pay, comes from the fact that the likelihood ratio

$$\ell(t,x) := \frac{p(t,x)}{q(x)} = p(t,x)\, e^{2\,\Psi(x)}\,, \qquad x \in \mathbb{R}^n\,,$$

which is of great importance in everything that goes on here, satisfies not a forward but a **backward** Kolmogorov equation:

$$\partial_t \ell(t,x) \;=\; \frac{1}{2}\,\Delta\ell(t,x) - \big\langle \nabla\ell(t,x), \nabla\Psi(x)\big\rangle_{\mathbb{R}^n}\,.$$

We fall back now on a long line that stretches back to Schrödinger (1931), Kolmogorov (1937) and passes, via Nelson (1966), to Föllmer (1985, 86), Haussmann & Pardoux (1986), and finally Meyer (1994).

So we fix $T \in (t_0, \infty)$, and look at the time-reversed process $X(t-s),\, 0 \le s \le T$ and at the filtration it generates:

$$\mathcal{G}(T-s) := \sigma\big(X(T-u),\, 0 \le u \le s\big)\,, \qquad 0 \le s \le T.$$

• *How does this look like? Is it (strongly) Markovian? is it a semimartingale? is it a diffusion?*

It stands to reason that, if you want to go backwards from where you are, you'd better know how you got there in the first place. Of course, we have known this since the time of Ariadne and Perseus; here, it means we have to know the transition probabilities.

. Turns out, Ariadne's thread is given here by the gradient of the log-transition-probability-density function — which acts here as an additive component to the potential:

$$\mathrm{d}X(T-s) = \nabla\Big(\log p(T-s,\cdot) + \Psi\Big)\big(X(T-s)\big)\mathrm{d}s + \mathrm{d}\overline{W}(T-s)$$

$$= \nabla\Big(\log \ell\big(T-s, X(T-s)\big) - \Psi\big(X(T-s)\big)\Big)\mathrm{d}s + \mathrm{d}\overline{W}(T-s)$$

where $\big(\overline{W}(T-s)\big)_{0\le s\le T}$ is a $\mathbb{P}-$Brownian motion of the time-reversed filtration.

THEOREM: de Bruijn Backwards Martingale.

Consider the cumulative Fisher information process

$$F(T-s) := \frac{1}{2}\int_0^s \left|\nabla \log \ell(T-u, X(T-u))\right|^2 \mathrm{d}u\,, \quad 0 \le s \le T$$

from the right. Then $\mathbb{E}^{\mathbb{P}}(F(0)) < \infty$, *and the process*

$$M(T-s) := \log\left(\frac{\ell(T-s, X(T-s))}{\ell(T, X(T))}\right) - F(T-s)$$

$$= \int_0^s \left\langle \nabla \log \ell(T-u, X(T-u)), \mathrm{d}\overline{W}(T-u)\right\rangle_{\mathbb{R}^n}$$

is a square-integrable $\mathbb{P}-$*martingale of the backwards filtration, with* $\langle M \rangle = 2F$.

The de Bruijn dissipation of relative entropy identity

$$\begin{aligned} H(P(t)|Q) - H(P(t_0)|Q) &= \mathbb{E}^{\mathbb{P}}\left[\log\left(\frac{\ell(t, X(t))}{\ell(t_0, X(t_0))}\right)\right] \\ &= -\frac{1}{2}\int_{t_0}^{t} I(P(\theta)|Q)\,\mathrm{d}\theta \end{aligned}$$

for $t \geq t_0$ follows from this right away, just by taking expectations.

TIME REVERSAL: THE PERTURBED CASE

In a completely analogous manner, we carry out this analysis also in the "perturbed" case.

Turns out, we have the dynamics

$$\mathrm{d}X(T-s) = \nabla\big(\log p^{\beta} + \Psi + B\big)(T-s, X(T-s))\mathrm{d}s + \mathrm{d}\overline{W}^{\beta}(T-s)$$

$$= \nabla\big(\log \ell^{\beta} - \Psi + B\big)(T-s, X(T-s))\mathrm{d}s + \mathrm{d}\overline{W}^{\beta}(T-s)$$

with $\big(\overline{W}^{\beta}(T-s)\big)_{0\leq s\leq T}$ a $\mathbb{P}^{\beta}-$Brownian motion of the time-reversed filtration.

THEOREM: "Perturbed" de Bruijn Backwards Martingale. *Consider the Fisher information process for the perturbed dynamics, accumulated from the right*

$$F^{\beta}(T-s) \;:=\; \int_0^s \Big[\, \frac{1}{2}\big|\nabla \log \ell^{\beta}(T-u, X(T-u))\big|^2$$

$$+ \Big(\big\langle \beta, 2\Psi \big\rangle_{\mathbb{R}^n} - div(\beta) \Big) \big(X(T-u) \big) \Big] \, \mathrm{d}u \,,$$

for $0 \le s \le T - t_0$. *Then* $\mathbb{E}^{\mathbb{P}^{\beta}}(F(t_0)) < \infty$, *and the process*

$$M^{\beta}(T-s) \;:=\; \log \left(\frac{\ell^{\beta}(T-s, X(T-s))}{\ell^{\beta}(T, X(T))} \right) - F^{\beta}(T-s)$$

$$= \int_0^s \Big\langle \nabla \log \ell^{\beta}(T-u, X(T-u)), \mathrm{d}\overline{W}^{\beta}(T-u) \Big\rangle_{\mathbb{R}^n}$$

is a square-integrable $\mathbb{P}^{\beta}-$*martingale of the backwards filtration, with* $\langle M^{\beta} \rangle = 2F^{\beta}$.

Once again, the "perturbed" de Bruijn identity

$$H(P^{\beta}(t)|Q) - H(P^{\beta}(t_0)|Q) = \mathbb{E}^{\mathbb{P}^{\beta}}\left[\log\left(\frac{\ell^{\beta}(t, X(t))}{\ell^{\beta}(t_0, X(t_0))}\right)\right]$$

$$= \int_{t_0}^{t} \mathbb{E}^{\mathbb{P}^{\beta}}\Big(-\frac{1}{2}\big|\nabla \log \ell^{\beta}(\theta, X(\theta))\big|^2 +$$

$$+ \Big(\operatorname{div}(\beta) - \langle \beta, 2\nabla\Psi\rangle_{\mathbb{R}^n}\Big)(X(\theta))\Big)\mathrm{d}\theta$$

for $t \geq t_0$ follows from this right away, just by taking expectations.

. We obtain now the perturbed (HI) identity, simply dividing by $t - t_0$, letting $t \downarrow t_0$, and integrating by parts.

TECHNICAL WORK

Lots of technical details are here under the rug. The indicated derivatives exist only outside a (countable, at most) set of exceptional points.

Very delicate analysis is necessary, in order to show that the exceptional set for the temporal dissipation of relative entropy is the same for the perturbed case, as for the unperturbed.

And even more delicate analysis is necessary, in order to show that the above set is also the exceptional set for the temporal growth of the Wasserstein distance along the Fokker-Planck flow — both unperturbed, and perturbed.

TRAJECTORIAL RATES OF RELATIVE ENTROPY DISSIPATION

With $0 < t_0 < T - s < T$, we have:

$$\lim_{s \uparrow T - t_0} \frac{1}{T - t_0 - s} \mathbb{E}^{\mathbb{P}} \left[\log \left(\frac{\ell(T-s, X(T-s))}{\ell(t_0, X(t_0))} \right) \Big| \mathcal{G}(T-s) \right] = -\frac{1}{2} \left| \nabla \log \ell \big(t_0, X(t_0)\big) \right|^2 ,$$

$$\lim_{s \uparrow T - t_0} \frac{1}{T - t_0 - s} \log \left(\frac{\ell^{\beta}(T-s, X(T-s))}{\ell\,(T-s, X(T-s))} \right) = \Big(\operatorname{div}(\beta) + \big\langle \beta, \nabla \log p(t_0, \cdot) \big\rangle_{\mathbb{R}^n} \Big) \big(X(t_0)\big) .$$

$$\lim_{s \uparrow T - t_0} \frac{1}{T - t_0 - s} \mathbb{E}^{\mathbb{P}} \left[\log \left(\frac{\ell^{\beta}(T - s, X(T - s))}{\ell^{\beta}(t_0, X(t_0))} \right) \Big| \mathcal{G}(T - s) \right]$$

$$= \Big(\mathrm{div}(\beta) + \langle \beta, \nabla \log p(t_0, \cdot) \rangle_{\mathbb{R}^n} \Big) (X(t_0)) - \frac{1}{2} \left| \nabla \log \ell(t_0, X(t_0)) \right|^2 ,$$

$$\lim_{s \uparrow T - t_0} \frac{1}{T - t_0 - s} \mathbb{E}^{\mathbb{P}^{\beta}} \left[\log \left(\frac{\ell^{\beta}(T - s, X(T - s))}{\ell^{\beta}(t_0, X(t_0))} \right) \Big| \mathcal{G}(T - s) \right]$$

$$= \Big(\mathrm{div}(\beta) - \langle \beta, \nabla \log p(t_0, \cdot) \rangle_{\mathbb{R}^n} \Big) (X(t_0)) - \frac{1}{2} \left| \nabla \log \ell(t_0, X(t_0)) \right|^2 ,$$

A SECOND EYE-OPENER: IN A SPECIAL CASE

Suppose $q(x) = e^{-2\Psi(x)}$ is a probability density function.

Imagine starting the Langevin-Smoluchowski diffusion

$$\mathrm{d}X(t) = -\nabla\Psi\big(X(t)\big)\,\mathrm{d}t + \mathrm{d}W(t)$$

with $X(0)$ — thus also $X(t)$ for all $t > 0$ — having this invariant density, and denote as $\mathbb{Q}$ the probability measure induced on $C\big([0,\infty) : \mathbb{R}^n\big)$ by the continuous process $X(\cdot)$.

. Then the time-reversed likelihood process

$$\ell\big(T-s, X(T-s)\big), \quad 0 \leq s \leq T \qquad \text{is a } \mathbb{Q}-\text{martingale}$$

of the time-reversed filtration (Fontbona & Jourdain (2016)), for any given $T \in (0,\infty)$.

And the decrease of the relative entropy

$$
\begin{aligned}
H\big(P(T)\big|Q\big) &= \mathbb{E}^{\mathbb{P}}\big[\log \ell\big(T, X(T)\big)\big] \\
&= \mathbb{E}^{\mathbb{Q}}\big[\ell\big(T, X(T)\big) \log \ell\big(T, X(T)\big)\big]
\end{aligned}
$$

is a direct consequence of the Jensen inequality.

THE HWI INEQUALITY IN THE "CONVEX" CASE

Let us suppose from now onward that $Q \in \mathcal{P}_2(\mathbb{R}^n)$, and that the potential satisfies the convexity condition

$$D^2\Psi(x) \geq \kappa \, \mathbf{Id} \, , \quad \forall \;\; x \in \mathbb{R}^n$$

for some real number κ. We claim that the perturbed (HI) identity we established in this case a few slides back contains, in seminal form, the celebrated Otto-Villani (2000) **HWI inequality**

$$H(P_0|Q) - H(P_1|Q) \leq W(P_0, P_1) \sqrt{I(P_0|Q)} - (\kappa/2) \, W^2(P_0, P_1).$$

. When $\kappa > 0$, this inequality leads to the Talagrand (1996) inequality, to the Gross (1974) Log-Sobolev inequality, to the Poincaré inequality, and to the exponential dissipation of relative entropy

$$H\big(P(t)|Q\big) \leq H\big(P(0)|Q\big) \, e^{-\kappa t} \, .$$

Pick two probability measures P_0, P_1 in the Wasserstein space $\mathcal{P}_2(\mathbb{R}^n)$, with compactly supported densities to make life simple (though ultimately not necessary).

Transport P_0 to P_1 by means of a **constant-speed geodesic** $(P_t)_{0\le t\le 1}$, as follows: for some convex $G : \mathbb{R}^n \to \mathbb{R}$,

$$W^2(P_0, P_1) = \int_{\mathbb{R}^n} \left| x - \nabla G(x) \right|^2 P_0(\mathrm{d}x) = \|\gamma\|^2_{L^2(P_0)}$$

by Brenier (1991), and $W(P_0, P_t) = t \cdot \|\gamma\|_{L^2(P_0)}$, $0 \le t \le 1$ for

$$P_t := \left(T_t^\gamma\right)_\# P_0 \, ; \qquad T_t^\gamma(x) := t \cdot \nabla G(x) + \left(1 - t\right) \cdot x \, , \quad x \in \mathbb{R}^n \, .$$

Denote the density function of this probability measure P_t by $p_t(\cdot)$, and define the likelihood ratio

$$\ell_t(x) := \frac{p_t(x)}{q(x)} = p_t(x) \, e^{2\Psi(x)} \, , \qquad x \in \mathbb{R}^n \, .$$

Then, by complete analogy with the perturbed (HI) identity

$$\lim_{t \downarrow t_0} \frac{H\big(P^\beta(t) \big| Q\big) - H\big(P(t_0) \big| Q\big)}{t - t_0} = \big\langle\big\langle\, \mathfrak{a}, \mathfrak{c} \,\big\rangle\big\rangle_{L^2(\mathbb{P})} ,$$

$$\mathfrak{a} = \nabla \log \ell\big(t_0, X(t_0)\big) , \qquad \mathfrak{b} = \beta\big(X(t_0)\big), \qquad \mathfrak{c} = -\frac{1}{2}\,\mathfrak{a} - \mathfrak{b}$$

already discussed, and identifying $P(0) = P_0$, $t_0 = 0$, $\ell(0, \cdot) = \ell_0$:

$$\lim_{t \downarrow 0} \frac{H\big(P_t \big| Q\big) - H\big(P_0 \big| Q\big)}{t} = \big\langle\big\langle\, \nabla \log \ell_0 \,, \gamma \,\big\rangle\big\rangle_{L^2(P_0)} .$$

These two "slopes" – one along the (curved, perturbed Fokker-Planck) flow $\big(P^\beta(t)\big)_{t \geq 0}$, the other along the straight flow (or constant-speed geodesic) $\big(P_t\big)_{0 \leq t \leq 1}$ – are EXACTLY THE SAME, if we select β via

$$\gamma = -\frac{1}{2}\,\nabla \log \ell_0 - \beta\,.$$

Now write the Taylor expansion

$$h(1) \;=\; h(0) + h'(0+) + \int_0^1 \big(1-t\big)\, h''(t)\,\mathrm{d}t$$

for this function $h(t) := H\big(P_t \big| Q\big)$. From the above computation and Cauchy-Schwarz, we get

$$\begin{aligned} h'(0+) \;=\; \big\langle\big\langle\, \nabla \log \ell_0\,, \gamma\,\big\rangle\big\rangle_{L^2(P_0)} \;&\geq\; -\big\|\nabla \log \ell_0\,\big\|_{L^2(P_0)} \cdot \big\|\gamma\,\big\|_{L^2(P_0)} \\ &= -\sqrt{I(P_0|Q)} \cdot W(P_0, P_1). \end{aligned}$$

On the other hand, McCann's (1994) displacement convexity results give

$$h''(t) \;\geq\; \kappa \cdot W^2\big(P_0, P_1\big)\,, \qquad 0 \leq t \leq 1$$

and the HWI inequality follows.

EPILOGUE

We are working on exploring the applicability of this trajectorial methodology to more general settings.

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HAPPY BIRTHDAY, NICOLE !

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