On the “blue death” and Poisson measure

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Population ageing and uncertainty around future longevity development:

- Sustainability of public pension systems, longevity risk management for insurers and pension fund...

**Diverging trends** in longevity: increasing of socioeconomic and geographical gaps in health and mortality.

- “Unfair” redistribution properties of pensions systems, errors in funding of annuity and pension obligation...

**Standard tools**: Data and mortality rates models

Decisive for evaluating pension reforms, computation of regulatory capital, pricing longevity financial products...
Challenges

- Observed mortality is a **by-product of the population dynamics** (not taken into account in standard models):
  - Result of complex demographic and social mechanisms.

**Limitations** of only studying age-specific mortality rates:

1. Relationship between **macro environment** and longevity improvement.
2. Understanding the impact of the population and its heterogeneity:
   - Aggregation issues.
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  - Ex: water purification in the 1900-30s, campaigns against smoking...

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The example of cholera epidemics in England and France over the 19th century
Modeling the macro environment

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- Cholera: waterborne acute infection nicknamed “blue-death”.
- Four major outbreaks in Europe over the 19th century (1830-90) causing
  - Tens of thousands of deaths.
  - In 1832: Death of 2% of Paris inhabitants.
Modeling the macro environment

- Intensity crisis ⇒ catalyst of public health measures
  - First regulation of water supply companies, development of sewage systems, health education...
  - First international meeting of health.

- Different consequences depending on political context and population sizes.

- Delayed impact on Mortality
  - More than 10 international conferences over 50 years (1851-1903) to take action.
  - Later (but major) impact on the increase in the duration of life.

Impact of economic growth

- Bidirectional (and controversial) relationship between mortality decline and economic growth.
Goal: study stochastic heterogeneous population dynamics including

- General random environment,
- Important changes of composition.

Model

- Population process $Z = (Z^i)_{i=1..p}$ structured by discrete subgroups filtration.
- Population evolves according to demographic events (birth, death) or changes of characteristics (swap).
- Random environment $\Rightarrow$ stochastic event intensities:

$$P(\text{ ev of type } \gamma \in ]t, t + dt][G_t]) \approx \mu^\gamma(\omega, t, Z_t)dt.$$
Jump measure

Standard framework:

- Continuous Time Markov Chain: \( P( \text{ev } \gamma \in ]t, t + dt] | G_t) \approx \mu(\gamma(Z_t)) dt. \)
- Extension when \( \mu(t, \cdot) = (\mu(\gamma(t, \cdot))) \) is \( G_0 \)-measurable.
- Event intensities characterize the distribution of the population process (martingale problem viewpoint).
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This is not the case here

- Each type of event (birth, death, swap) \( \gamma \in \mathcal{J} \) is associated with the counting process:
  \[
  N_t^\gamma = \sum_{0<s\leq t} 1_{\{\Delta Z_s = \phi(\gamma)\}}
  \]
  (1)
- Pathwise representation of the jump measure \( \mathbf{N} = (N^\gamma)_{\gamma \in \mathcal{J}} \) (in the spirit of Bremaud Massoulié (1998))
Pathwise representation and strong domination

- **Thinning** and projection of extended Poisson measure on random sets:

\[
N_t = \int_0^t \int_{\mathbb{R}^+} 1_{\{\theta \leq \mu(s, Z_{s^-})\}} \mathbb{Q}(ds, d\theta), \quad Z_t = F(Z_0, N_t). \quad (2)
\]
Pathwise representation and strong domination

- **Thinning** and projection of extended Poisson measure on random sets:

\[ N_t = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leq \mu(s, Z_{s-})\}} Q(ds, d\theta), \quad Z_t = F(Z_0, N_t). \quad (2) \]

- **Existence of non-explosive solutions**: control birth intensities

\[ \mu^b(\omega, t, z) \leq k_t g(z^b), \quad (3) \]

with \( g \) verifying \( \sum_{n \geq 1} \frac{1}{\sum g^i(n)} = \infty. \)

**Proposition**

*There exists a unique well-defined solution \( N \) of (2), strongly dominated by a multivariate counting process \( G \): \( G - N \) is a multivariate counting process.*
Population with two time-scales

Study of the population evolution when composition changes occurs at a fast pace in comparison with the demographic scale.

- Also applications in ecology (ex: viability in patchy environment).
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- Hyp: intensity of swap events $\sim O\left(\frac{1}{\epsilon}\right) \gg$ demographic events $\sim O(1)$

$$dN_{t,\epsilon}^{s} = Q^{s}(dt, [0, \frac{1}{\epsilon} \mu^{s}(t, Z_{t-}^{\epsilon})]), \quad dN_{t,\epsilon}^{\text{dem}} = Q^{\text{dem}}(dt, [0, \mu^{\text{dem}}(t, Z_{t-}^{\epsilon})]).$$
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$$dN_{t}^{s,\epsilon} = Q^{s}(dt, [0, \frac{1}{\epsilon}\mu^{s}(t, Z_{t-}^{\epsilon})]), \quad dN_{t}^{dem,\epsilon} = Q^{dem}(dt, [0, \mu^{dem}(t, Z_{t-}^{\epsilon})]).$$

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What can we say about the aggregated population?

$$Z_{t}^{\hat{a},\epsilon} = \sum_{i=1}^{p} Z_{t}^{i} = F(Z_{0}, N_{t}^{dem,\epsilon}),$$

Aggregated birth and death intensities:

$$\mu^{b,\hat{a}}(t, Z_{t}) = \sum_{i=1}^{p} \mu^{b,i}(t, Z_{t}), \quad \mu^{d,\hat{a}}(t, Z_{t}) = \sum_{i=1}^{p} \mu^{d,i}(t, Z_{t})$$
Pure swap processes $X$ of $G_t$-intensity $\mu^s$:

- Population with NO demographic events.
- Constant size: $X_0^b = d \Rightarrow X_t = d$ ($X \in \mathcal{U}_d$, populations of size $d$).

$$f(X_t) - \int_0^t \sum_{i \neq j} (f(X_s + e_j - e_i) - f(X_s)) \mu^{(i,j)}(s, X_s) ds$$

is a $(G_t)$ (local) martingale.

**Assumption** There exists a unique $\mathcal{O} \otimes \mathcal{P}(\mathbb{N})$-measurable random kernel $(\Gamma(\omega, t, n, dz))$ such that $\forall n \geq 0$, $t \geq 0$ and $f : \mathbb{N}^p \mapsto \mathbb{R}$:

$$\Gamma(t, n, L_{t}^{\text{sw}} f \mathbb{1}_{\mathcal{U}_n}) = \int_{\mathcal{U}_n} L_{t}^{\text{sw}} f(z) \Gamma(t, n, dz) = 0.$$  \hspace{1cm} (4)
Theorem (partial)

Under Assumption 4, the aggregated processes $Z^{\varepsilon,\bar{n}}$ converge in distribution to a BD process $\bar{Z}^{\bar{n}}$ of intensity:

$$
\lambda^b(t, \bar{Z}_t^{\bar{n}}) = \int_{U_{\bar{Z}_t^{\bar{n}}}} \mu^{b,\bar{n}}(t, z) \Gamma(t, \bar{Z}_t^{\bar{n}}, dz), \quad \lambda^d(t, \bar{Z}_t^{\bar{n}}) = \int_{U_{\bar{Z}_t^{\bar{n}}}} \mu^{d,\bar{n}}(t, z) \Gamma(t, \bar{Z}_t^{\bar{n}}, dz).
$$

- Main problem: stochastic intensity functional $\mu(\omega, t, \cdot)$ (no regularity)
  $\Rightarrow$ extension of existing results (Kurtz (1992), Yin and Zhang (2004)).
- Tools: strong domination construction and stable convergence.
Thank you for your attention ...
Thank you for your attention ... and Happy birthday Nicole (and please be on time)!