On the “blue death” and Poisson measure

Sarah Kaakaï

Laboratoire Manceau de Mathématiques, Le Mans Université

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Longevity risk and heterogeneity

- Population ageing and uncertainty around future longevity development:
  - ↳ Sustainability of public pension systems, longevity risk management for insurers and pension fund...
- **Diverging trends** in longevity: increasing of socioeconomic and geographical gaps in health and mortality.
  - ↳ "Unfair" redistribution properties of pensions systems, errors in funding of annuity and pension obligation...

- **Standard tools**: Data and mortality rates models
  Decisive for evaluating pension reforms, computation of regulatory capital, pricing longevity financial products...

Challenges

- Observed mortality is a by-product of the population dynamics (not taken into account in standard models):

  ↳ Result of complex demographic and social mechanisms.

Limitations of only studying age-specific mortality rates:

1. Relationship between macro environment and longevity improvement.
2. Understanding the impact of the population and its heterogeneity:

   ↳ Aggregation issues.

Modeling the macro environment

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- Cholera: waterborne acute infection nicknamed "blue-death".
- Four major outbreaks in Europe over the 19th century (1830-90) causing
  - Tens of thousands of deaths.
  - In 1832: Death of 2% of Paris inhabitants.

Modeling the macro environment

- Intensity crisis $\Rightarrow$ catalyst of public health measures
  - First regulation of water supply companies, development of sewage systems, health education...
  - First international meeting of health.
- Different consequences depending on political context and population sizes.
- Delayed impact on Mortality
  - More than 10 international conferences over 50 years (1851-1903) to take action.
  - Later (but major) impact on the increase in the duration of life.

Impact of economic growth

- Bidirectional (and controversial) relationship between mortality decline and economic growth.

Birth Death Swap model

Goal: study stochastic heterogeneous population dynamics including

- General random environment,
- Important changes of composition.

Model

- Population process $Z = (Z^i)_{i=1..p}$ structured by discrete subgroups filtration.
- Population evolves according to demographic events (birth, death) or changes of characteristics (swap).
- Random environment $\Rightarrow$ stochastic event intensities:

$$\mathrm{P}(\text{ ev of type } \gamma \in]t, t + \mathrm{d}t] | \mathcal{G}_t) \simeq \mu^\gamma(\omega, t, Z_t)\mathrm{d}t.$$

Jump measure

Standard framework:

- Continuous Time Markov Chain: P(ev $\gamma \in]t, t + \mathrm{d}t] | \mathcal{G}_t) \simeq \mu^\gamma(Z_t)\mathrm{d}t$.
- Extension when $\boldsymbol{\mu}(t, \cdot) = (\mu^\gamma(t, \cdot))$ is $\mathcal{G}_0$-measurable.
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This is not the case here

- Each type of event (birth, death, swap) $\gamma \in \mathcal{J}$ is associated with the counting process:

$$N_t^\gamma = \sum_{0 < s \leqslant t} \mathbb{1}_{\{\Delta Z_s = \phi(\gamma)\}} \tag{1}$$

- Pathwise representation of the jump measure $\mathbf{N} = (N^\gamma)_{\gamma \in \mathcal{J}}$ (in the spirit of Bremaud Massoulié (1998))

Pathwise representation and strong domination

- Thinning and projection of extended Poisson measure on random sets:

$$\mathbf{N}_t = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leqslant \boldsymbol{\mu}(s, Z_{s-})\}} \mathbf{Q}(\mathrm{d}s, \mathrm{d}\theta), \quad Z_t = F(Z_0, \mathbf{N}_t). \qquad (2)$$

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- **Existence of non-explosive solutions**: control birth intensities

$$\boldsymbol{\mu}^b(\omega, t, z) \leqslant k_t \mathbf{g}(z^\natural), \tag{3}$$

with $\mathbf{g}$ verifying $\sum_{n \geqslant 1} \frac{1}{\sum g^i(n)} = \infty$.

Proposition

There exists a unique well-defined solution $\mathbf{N}$ *of* (2), **strongly dominated** *by a multivariate counting process* $\mathbf{G}$*:* $\mathbf{G} - \mathbf{N}$ *is a multivariate counting process.*

Population with two time-scales

Study of the population evolution when composition changes occurs at a fast pace in comparison with the demographic scale.

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- **Hyp**: intensity of swap events $\sim O(\frac{1}{\epsilon}) >>$ demographic events $\sim O(1)$

$$d\mathbf{N}_t^{s,\epsilon} = \mathbf{Q}^s(\mathrm{d}t, [0, \frac{1}{\epsilon}\boldsymbol{\mu}^s(t, Z_{t-}^\epsilon)]), \quad d\mathbf{N}_t^{\mathrm{dem},\epsilon} = \mathbf{Q}^{\mathrm{dem}}(\mathrm{d}t, [0, \boldsymbol{\mu}^{\mathrm{dem}}(t, Z_{t-}^\epsilon)]).$$

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What can we say about the aggregated population?

$$Z_t^{\natural,\epsilon} = \sum_{i=1}^{p} Z_t^i = F(Z_0, \mathbf{N}_t^{\mathrm{dem},\epsilon}),$$

Aggregated birth and death intensities:

$$\mu^{b,\natural}(t, Z_t) = \sum_{i=1}^{p} \mu^{b,i}(t, Z_t), \quad \mu^{d,\natural}(t, Z_t) = \sum_{i=1}^{p} \mu^{d,i}(t, Z_t)$$

Pure Swap process

Pure swap processes X of $\mathcal{G}_t$-intensity $\boldsymbol{\mu}^s$:

- Population with NO demographic events.
- Constant size: $X_0^\natural = d \Rightarrow X_t = d$ ($X \in \mathcal{U}_d$, populations of size d).

$$f(X_t) - \int_0^t \underbrace{\sum_{i \neq j} \big(f(X_s + \mathbf{e}_j - \mathbf{e}_i) - f(X_s)\big)\mu^{(i,j)}(s, X_s)}_{L_s^{\mathrm{sw}} f(X_s)} \mathrm{d}s \text{ is a } (\mathcal{G}_t) \text{ (local) martingale.}$$

Assumption There exists a unique $\mathcal{O} \otimes \mathcal{P}(\mathbb{N})$-measurable random kernel $(\Gamma(\omega, t, n, \mathrm{d}z))$ such that $\forall n \geqslant 0$, $t \geqslant 0$ and $f : \mathbb{N}^p \mapsto \mathbb{R}$:

$$\Gamma(t, n, L_t^{\mathrm{sw}} f \mathbb{1}_{\mathcal{U}_n}) = \int_{\mathcal{U}_n} L_t^{\mathrm{sw}} f(z) \Gamma(t, n, \mathrm{d}z) = 0. \tag{4}$$

Averaging result

Theorem (partial)

Under Assumption 4, the **aggregated processes** $Z^{\epsilon,\natural}$ *converge in distribution to a BD process* $\bar{Z}^{\natural}$ *of intensity:*

$$\lambda^b(t, \bar{Z}_t^{\natural}) = \int_{\mathcal{U}_{\bar{Z}_t^{\natural}}} \mu^{b,\natural}(t,z)\Gamma(t, \bar{Z}_t^{\natural}, \mathrm{d}z), \quad \lambda^d(t, \bar{Z}_t^{\natural}) = \int_{\mathcal{U}_{\bar{Z}_t^{\natural}}} \mu^{d,\natural}(t,z)\Gamma(t, \bar{Z}_t^{\natural}, \mathrm{d}z).$$

- Main problem: stochastic intensity functional $\boldsymbol{\mu}(\omega, t, \cdot)$ (no regularity) $\Rightarrow$ extension of existing results (Kurtz (1992), Yin and Zhang (2004)).
- Tools: strong domination construction and stable convergence.

Thank you for you attention ...

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and Happy birthday Nicole (and please be on time) !