Models of default times

Monique Jeanblanc joint work with Libo LI, and Yiqing LIM, Liming YIN
LaMME, University Paris Saclay, UNSW; Jiao Tong Shanghai

Nicole’s Birthday; Jussieu, 24th May 2019
Default time

- Given a measurable filtered probability space \((\Omega, \mathcal{F}, \mathbb{F})\), a default time is a positive random variable.
- The default process is \(A_t = 1_{\{\tau \leq t\}}\). Denoting by \(\mathbb{A}\) the filtration generated by \(A\), the filtration \(\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}\) is the smallest filtration containing \(\mathbb{F}\) and \(\mathbb{A}\).
- We denote by \(A^p\) (resp. \(A^o\)) the dual \(\mathbb{F}\)-predictable (resp. optional) projection of \(A\). We denote by \(\mathcal{J}^o\) (resp. \(\mathcal{J}^p\)) the set of jump times of \(A^o\) (resp. \(A^p\)).
• We denote by $A^p$ (resp. $A^o$) the dual $\mathbb{F}$-predictable (resp. optional) projection of $A$. We denote by $\mathcal{J}^o$ (resp. $\mathcal{J}^p$) the set of jump times of $A^o$ (resp. $A^p$).

• We denote by $\mathcal{A}$ the set of $\mathbb{F}$-stopping times $\vartheta$ such that $\mathbb{P}(\tau = \vartheta) > 0$. Then, $\mathcal{A} = \mathcal{J}^o$ and $\mathcal{J}^p$ is the subset of $\mathcal{A}$ made of predictable stopping times. In particular, $\tau$ avoids $\mathbb{F}$ stopping times if and only if $A^o$ is continuous.

• The process $Z$ defined as $Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$, called the Azéma supermartingale admits a Doob-Meyer decomposition $Z = M - A^p$. 
Intensity process

- The compensator of $A$ or the **intensity process** of $\tau$ is the increasing $\mathcal{G}$-predictable process $\Lambda$ such that $A - \Lambda$ is a $\mathcal{G}$-martingale, given by $\Lambda_t = \int_0^{t \wedge \tau} \frac{dA^p_s}{Z_s^-}$.

- In case where $Z > 0$, the process $Z$ admits a unique multiplicative decomposition $Z = Ne^{-\Gamma}$ where $N$ is a local $\mathcal{F}$-martingale and $\Gamma$ an increasing $\mathcal{F}$-predictable process. If $\Gamma$ is continuous, the intensity process is $\Gamma^\tau$.

- It follows that the intensity process does not contain full information about $Z$. 

Simple Defaultable claims

- Payment at maturity and recovery at hit.

\[ \zeta := Y_T 1_{\{\tau > T\}} + C_{\tau} 1_{\{\tau \leq T\}} \]

where \( Y_T \in \mathcal{F}_T \) and \( C \) is an \( \mathbb{F} \)-predictable process. One has

\[ \mathbb{E}[\zeta \mid \mathcal{G}_t] = Z_t^{-1} \mathbb{E}[Y_T Z_T + \int_{(t,T]} C_u dA^P_u \mid \mathcal{F}_t] 1_{\{\tau > t\}} + C_{\tau} 1_{\{\tau \leq t\}} \]

- In the case where \( C \) is \( \mathbb{F} \)-optional, then one has to replace \( A^P \) with \( A^o \) in the above formula.
A second Azéma supermartingale

- The supermartingale $Z$ does not contain full information about $A^o$: it is not possible to recover $A^o$ from $Z$.
- The second Azéma supermartingale is

$$
\tilde{Z}_t := \mathbb{P}(\tau \geq t | \mathcal{F}_t).
$$

This supermartingale is lâdlàg and admits a unique Doob-Meyer-Mertens decomposition $\tilde{Z} = m - A^o$ where $m$ is an $\mathbb{F}$-martingale.
- The processes $Z, \tilde{Z}, A^o, A^p$ and $\Lambda$ play an important role. Note that $Z_t = \mathbb{E}[A^p_\infty - A^p_t | \mathcal{F}_t]$ and $\tilde{Z}_t = \mathbb{E}[A^o_\infty - A^o_t | \mathcal{F}_t]$. 

Revisiting Cox’s model

Let $K$ be an increasing $\mathbb{F}$-adapted process with $K_0 = 0$ and

$$\tau = \inf\{t : K_t \geq \Theta\}$$

where $\Theta$ is an exponential r.v. independent from $\mathbb{F}$. Then $\mathbb{F}$ is immersed in $\mathbb{G}$.

Let $\mathcal{J}(K)$ the set of jump times of $K$, then $\mathcal{A} = \mathcal{J}(K)$.

- If $K$ is continuous, one has

$$Z = e^{-K} = \tilde{Z}, \ A^o = A^p = 1 - Z, \ \Lambda = K,$$

- If $K$ is predictable and continuous on right,

$$Z = e^{-K}, \ A^p = 1 - Z, \ \Lambda = K_\to - \sum(e^{-\Delta K} - 1)$$

- If $K$ is continuous on right,

$$Z = e^{-K}, \ \tilde{Z} = e^{-K_\to}, \ A^o = 1 - e^{-K},$$
• If $K$ is continuous on left, one has
\[ \tilde{Z} = e^{-K}, \quad Z = \tilde{Z}_+ . \]

• Therefore, one can construct default times with a given set $\mathcal{A}$. Jiao and Li, Gehmlich and Schmidt have produced models where $\mathcal{A}$ is not empty.

• Any random time admits a unique decomposition as $\xi \wedge \vartheta$ where $\xi$ avoids stopping times and $\vartheta$ is thin and $\xi \vee \vartheta = \infty$. 
Another characteristic is the conditional cumulative distribution

\[ F_t(u) = \mathbb{P}(\tau \leq u|\mathcal{F}_t). \]

It is known that if \( L(u) \) is a family of martingales, increasing w.r.t. \( u \), valued in \([0, 1]\) then, one can construct on an extended probability space a random time \( \tau \) (in fact the identity) and a probability \( \mathbb{Q} \) such that

- \( \forall t \geq 0, \mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t} \)
- \( \mathbb{Q}(\tau \leq u|\mathcal{F}_t) = L_t(u). \)
Density process

• The conditional law admits a density when
\[ \mathbb{E}[f(\tau)|\mathcal{F}_t] = \int_0^\infty f(u) p_t(u) \nu(du) \] where \( \nu \) is the law of \( \tau \).
Note that \( p(u) \) is a family of nonnegative martingales and that
\[ F_t(u) = \mathbb{P}(\tau \leq u|\mathcal{F}_t) = \int_0^u p_t(\theta) \nu(d\theta) \]
defines a family of martingales, valued in \([0,1]\), increasing w.r.t. \( u \).

• a family of density is a family of nonnegative martingales \( p(u) \) such that for any \( t \),
\[ \int_0^\infty p_t(u) \eta(du) = 1. \]

• Very few examples are known in the literature.
Density process

The density process is necessary in the two following cases

- Pricing of defaultable claims with payment at maturity: assume that the payoff, done at time $T$ is of the form $\zeta = f(\tau, Y_T)$.

\[
Z_t^{-1} \mathbb{E} \left[ \int_{(t,T]} f(u, Y_T) p_T(du) \big| \mathcal{F}_t \right].
\]

- Optimisation problems. In that case, one needs the decomposition of $\mathbb{F}$ martingales into $\mathbb{G}$ martingales which is

\[
n_t X_t = \hat{X}_t + \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\langle X, m \rangle_s^\mathbb{F} + \int_t^{t \wedge \tau} \frac{1}{p_{s-}(\tau)} d\langle X, p(u) \rangle_s^\mathbb{F} |_{u=\tau},
\]

where $\hat{X}$ is a $\mathbb{G}$-local martingale.
Construction of a random time from a $[0, 1]$ valued supermartingale

Assume that $Z$ is a continuous supermartingale valued in $]0, 1[$ with multiplicative decomposition $Z = Ne^{-\Lambda}$ where $\Lambda$ is continuous.

- We set, for $u < t$,

$$F_t(u) = (1 - Z_t) \exp \left\{ - \int_u^t \frac{Z_s}{1 - Z_s} d\Lambda_s \right\},$$

and for $t \leq u$

$$F_t(u) = \mathbb{E}[F_u(u)|\mathcal{F}_t] = \mathbb{E}[1 - Z_u|\mathcal{F}_t].$$

- The family $F(u)$ is a family of martingales, valued in $(0, 1]$, increasing w.r.t. $u$. 
• Note that, for $u < t$,

$$dF_t(u) = -\frac{F_t(u)}{1 - Z_t} dM_t$$

where $M$ is the martingale part in the Doob-Meyer decomposition of $Z$;

• Other solutions: let $Y$ be any continuous local martingale and $f$ be any bounded Lipschitz function with $f(0) = 0$, then a family $F(u)$ is given by

$$dF_t(u) = -F_t(u) \left( \frac{1}{1 - Z_t} dM_t + f(F_t(u) - 1 + Z_t) dY_t \right)$$

$$F_u(u) = 1 - Z_u$$
More generally, for $Z$ valued in $]0, 1[$, the family $F(u)$ defined as

$$dF_t(u) = -F_t(u) \frac{1}{1 - pZ_t} dM_t, \text{ for } t > u$$

$$F_u(u) = 1 - Z_u$$

$$F_t(u) = \mathbb{E}[1 - Z_u | \mathcal{F}_t], \text{ for } u > t$$

is a family of conditional laws. One can show that

$$F_t(u) = (1 - Z_t) \exp(- \int_{(u,t]} \frac{dA_{s}^{p,c}}{1 - pZ_s}) \prod_{u < s \leq t} \left(1 - \frac{\Delta A_{s}^{p}}{1 - pZ_s}\right)$$

where $pZ$ is the predictable projection of $Z$ and $A_{s}^{p,c}$ is the continuous part of $A^{p}$. 
Construction of densities

- If $F(u)$ is differentiable w.r.t. $u$, we obtain the conditional density of $\tau$.
- For example, in the previous case with $\Lambda_t = \int_0^t \lambda_s ds$

$$F_t(u) = (1 - Z_t) \exp \left\{ - \int_u^t \frac{Z_s}{1 - Z_s} \lambda_s ds \right\}$$

we obtain that, for $t > u$

$$g_t(u) = F_t(u) \frac{Z_u}{1 - Z_u} \lambda_u$$

and

$$dg_t(u) = \frac{Z_u}{1 - Z_u} \lambda_u dF_t(u) = g_t(u) \frac{1}{1 - Z_t} dM_t$$
Construction of supermartingales valued in $[0, 1]$

- **Starting from dual projections:** If $K$ is an $\mathbb{F}$-predictable increasing process such that $G_t := \mathbb{E}[K_\infty - K_t | \mathcal{F}_t] \leq 1$, then $G$ is a supermartingale valued in $[0, 1]$ and is the Azéma supermartingale of a random time, and $K$ will be the dual predictable projection of $A$.

- **Starting from dual projections:** If $K$ is an $\mathbb{F}$-optional increasing process such that $\tilde{G}_t := \mathbb{E}[K_\infty - K_t^- | \mathcal{F}_t] \leq 1$, then $\tilde{G}$ is a supermartingale valued in $[0, 1]$ and $K$ will be the dual optional projection of $A$. 

• **Starting with intensity rate:** Let $B$ be a Brownian motion. The solution of

$$dG_t = -\lambda_t G_t dt + b_t G_t (1 - G_t) dB_t, \quad G_0 = 1$$

where $\lambda \geq 0$ and $b$ are two bounded optional processes is a supermartingale valued in $[0, 1]$ and $\lambda$ is the intensity rate of the associated random times. The conditional density of $\tau$ is, for $t > u$

• For $t < u$, if $\lambda$ is deterministic, and setting $\Lambda_t = \int_0^t \lambda(s) ds$ and $N_t = G_t e^{\Lambda(t)}$

$$\mathbb{E}[G_u | \mathcal{F}_t] = e^{-\Lambda(u)} \mathbb{E}[N_u | \mathcal{F}_t] = N_t e^{-\Lambda(u)}$$

so that $g_t(u) = \lambda(u) e^{-\Lambda(u)} N_t$.

If follows that

$$g_t(u) = \lambda(u) e^{-\Lambda(u)} N_t \mathbf{1}_{\{t \leq u\}} + \lambda(u) e^{-\Lambda(u)} \frac{N_u}{\nu_u} \nu_t \mathbf{1}_{\{t > u\}}$$

where $d\nu_t = -b_t e^{-\Lambda(t)} \nu_t N_t dB_t$. 
For $\Gamma$ a continuous increasing $\mathbb{F}$ adapted process

$$dG_t = -G_t d\Gamma_t + b_t G_t (1 - G_t) dB_t, \ G_0 = 1$$

is a supermartingale valued in $[0, 1]$. This is in particular the case for last passage times, where $\Gamma$ involves a local time, or when $G = Y \wedge 1$ for a positive supermartingale $Y$. 
References: Generalities


References: Construction of default times


Jeanblanc, M. and Song, S., Explicit Model of Default Time with given Survival Probability, SPA, 121, 1678-170, 2011


Si on avait son anniversaire deux fois par an, on vivrait deux fois plus vieux.
Thank you for your attention