Models of default times

Monique Jeanblanc joint work with Libo LI, and Yiqing LIM, Liming YIN

LaMME, University Paris Saclay, UNSW; Jiao Tong Shanghai

Nicole's Birthday; Jussieu, 24th May 2019

Default time

- Given a measurable filtered probability space $(\Omega, \mathcal{F}, \mathbb{F})$, a default time is a positive random variable.
- The default process is $A_t = \mathbf{1}_{\{\tau \leq t\}}$. Denoting by $\mathbb{A}$ the filtration generated by A, the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is the smallest filtration containing $\mathbb{F}$ and $\mathbb{A}$.
- We denote by A^p (resp. A^o) the dual $\mathbb{F}$-predictable (resp. optional) projection of A. We denote by $\mathcal{J}^o$ (resp. $\mathcal{J}^p$) the set of jump times of A^o (resp. A^p).

- We denote by A^p (resp. A^o) the dual $\mathbb{F}$-predictable (resp. optional) projection of A. We denote by $\mathcal{J}^o$ (resp. $\mathcal{J}^p$) the set of jump times of A^o (resp. A^p).
- We denote by $\mathcal{A}$ the set of $\mathbb{F}$-stopping times ϑ such that $\mathbb{P}(\tau = \vartheta) > 0$. Then, $\mathcal{A} = \mathcal{J}^o$ and $\mathcal{J}^p$ is the subset of $\mathcal{A}$ made of predictable stopping times. In particular, τ avoids $\mathbb{F}$ stopping times if and only if A^o is continuous.
- The process Z defined as $Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$, called the Azéma supermartingale admits a Doob-Meyer decomposition $Z = M - A^p$.

Intensity process

- The compensator of A or the **intensity process** of τ is the increasing $\mathbb{G}$-predictable process Λ such that $A - \Lambda$ is a $\mathbb{G}$-martingale, given by $\Lambda_t = \int_0^{t\wedge\tau} \frac{dA^p_s}{Z_{s-}}$.
- In case where $Z > 0$, the process Z admits a unique multiplicative decomposition $Z = Ne^{-\Gamma}$ where N is a local $\mathbb{F}$-martingale and Γ an increasing $\mathbb{F}$-predictable process. If Γ is continuous, the intensity process is Γ^τ.
- It follows that the intensity process does not contain full information about Z.

Simple Defaultable claims

- Payment at maturity and recovery at hit.

$$\zeta := Y_T \mathbf{1}_{\{\tau > T\}} + C_\tau \mathbf{1}_{\{\tau \leq T\}}$$

where $Y_T \in \mathcal{F}_T$ and C is an $\mathbb{F}$-predictable process. On has

$$\mathbb{E}[\zeta \,|\, \mathcal{G}_t] = Z_t^{-1} \mathbb{E}[Y_T Z_T + \int_{(t,T]} C_u dA_u^p \,|\, \mathcal{F}_t] \mathbf{1}_{\{\tau > t\}} + C_\tau \mathbf{1}_{\{\tau \leq t\}}$$

- In the case where C is $\mathbb{F}$-optional, then one has to replace A^p with A^o in the above formula.

A second Azéma supermartingale

- The supermartingale Z does not contain full information about A^o: it is not possible to recover A^o from Z.
- The second Azéma supermartingale is

$$\widetilde{Z}_t := \mathbb{P}(\tau \geq t | \mathcal{F}_t) .$$

  This supermartingale is làdlàg and admits a unique Doob-Meyer-Mertens decomposition $\widetilde{Z} = m - A^o_-$ where m is an $\mathbb{F}$-martingale.
- The processes $Z, \widetilde{Z}$, A^o, A^p and Λ play an important role. Note that $Z_t = \mathbb{E}[A^p_\infty - A^p_t | \mathcal{F}_t]$ and $\widetilde{Z}_t = \mathbb{E}[A^o_\infty - A^o_{t-} | \mathcal{F}_t]$.

Revisiting Cox's model

Let K be an increasing $\mathbb{F}$-adapted process with $K_0 = 0$ and

$$\tau = \inf\{t : K_t \geq \Theta\}$$

where Θ is an exponential r.v. independent from $\mathbb{F}$. Then $\mathbb{F}$ is immersed in $\mathbb{G}$.

Let $\mathcal{J}(K)$ the set of jump times of K, then $\mathcal{A} = \mathcal{J}(K)$.

- If K is continuous, one has

$$Z = e^{-K} = \widetilde{Z},\ A^o = A^p = 1 - Z,\ \Lambda = K,$$

- If K is predictable and continuous on right,

$$Z = e^{-K},\ A^p = 1 - Z,\ \Lambda = K_- - \sum (e^{-\Delta K} - 1)$$

- If K is continuous on right,

$$Z = e^{-K},\ \widetilde{Z} = e^{-K_-},\ A^o = 1 - e^{-K},$$

- If K is continuous on left, one has

$$\widetilde{Z} = e^{-K}, \ Z = \widetilde{Z}_{+} \ .$$

- Therefore, on can construct default times with a given set $\mathcal{A}$. Jiao and Li, Gehmlich and Schmidt have produced models where $\mathcal{A}$ is not empty
- Any random time admits a unique decomposition as $\xi \wedge \vartheta$ where ξ avoids stopping times and ϑ is thin and $\xi \vee \vartheta = \infty$.

- Another characteristic is the conditional cumulative distribution
$$F_t(u) = \mathbb{P}(\tau \le u|\mathcal{F}_t)\,.$$
- It is known that if $L(u)$ is a family of martingales, increasing w.r.t. u, valued in $[0,1]$ then, one can construct on an extended probability space a random time τ (in fact the identity) and a probability $\mathbb{Q}$ such that
  - $\forall t \ge 0,\ \mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$
  - $\mathbb{Q}(\tau \le u|\mathcal{F}_t) = L_t(u)$.

Density process

- The conditional law admits a density when $\mathbb{E}[f(\tau)|\mathcal{F}_t] = \int_0^\infty f(u)p_t(u)\nu(du)$ where ν is the law of τ. Note that $p(u)$ is a family of nonnegative martingales and that

$$F_t(u) = \mathbb{P}(\tau \leq u|\mathcal{F}_t) = \int_0^u p_t(\theta)\nu(d\theta)$$

  defines a family of martingales, valued in $[0, 1]$, increasing w.r.t. u.
- a family of density is a family of nonnegative martingales $p(u)$ such that for any t, $\int_0^\infty p_t(u)\eta(du) = 1$.
- Very few examples are known in the literature.

Density process

The density process is necessary in the two following cases

- Pricing of defaultable claims with payment at maturity: assume that the payoff, done at time T is of the from $\zeta = f(\tau, Y_T)$.

$$Z_t^{-1}\mathbb{E}\Big[\int_{(t,T]} f(u, Y_T)\, p_T(du) \,|\, \mathcal{F}_t\Big]\,.$$

- Optimisation problems. In that case, one needs the decomposition of $\mathbb{F}$ martingales into $\mathbb{G}$ martingales which is

$$ntX_t = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s^{\mathbb{F}} + \int_{t\wedge\tau}^t \frac{1}{p_{s-}(\tau)} d\langle X, p(u)\rangle_s^{\mathbb{F}}{}_{|u=\tau},$$

where $\widehat{X}$ is a $\mathbb{G}$-local martingale.

Construction of a random time from a $[0,1]$ valued supermartingale

Assume that Z is a continuous supermartingale valued in $]0,1[$ with multiplicative decomposition $Z = Ne^{-\Lambda}$ where Λ is continuous.

- We set, for $u < t$,

$$F_t(u) = (1 - Z_t) \exp \left\{ - \int_u^t \frac{Z_s}{1 - Z_s} d\Lambda_s \right\} ,$$

  and for $t \leq u$

$$F_t(u) = \mathbb{E}[F_u(u)|\mathcal{F}_t] = \mathbb{E}[1 - Z_u|\mathcal{F}_t] .$$

- The family $F(u)$ is a family of martingales, valued in $(0,1]$, increasing w.r.t. u.

- Note that, for $u < t$,

$$dF_t(u) = -\frac{F_t(u)}{1 - Z_t} dM_t$$

where M is the martingale part in the Doob-Meyer decomposition of Z;

- Other solutions: let Y be any continuous local martingale and f be any bounded Lipschitz function with $f(0) = 0$, then a family $F(u)$ is given by

$$\begin{aligned} dF_t(u) &= -F_t(u)\left(\frac{1}{1 - Z_t} dM_t + f\big(F_t(u) - 1 + Z_t\big) dY_t\right) \\ F_u(u) &= 1 - Z_u \end{aligned}$$

More generally, for Z valued in $]0,1[$, the family $F(u)$ defined as

$$\begin{array}{rcl} dF_t(u) & = & -F_{t-}(u)\dfrac{1}{1-{}^pZ_t}dM_t, \text{ for } t > u \\ F_u(u) & = & 1 - Z_u \\ F_t(u) & = & \mathbb{E}[1 - Z_u|\mathcal{F}_t], \text{ for } u > t \end{array}$$

is a family of conditional laws. One can show that

$$F_t(u) = (1 - Z_t)\exp(-\int_{(u,t]} \frac{dA_s^{p,c}}{1-{}^pZ_s}) \prod_{u<s\le t} \big(1 - \frac{\Delta A_s^p}{1-{}^pZ_s}\big)$$

where ${}^pZ$ is the predictable projection of Z and $A^{p,c}$ is the continuous part of A^p.

Construction of densities

- If $F(u)$ is differentiable w.r.t. u, we obtain the conditional density of τ.
- For example, in the previous case with $\Lambda_t = \int_0^t \lambda_s ds$

$$F_t(u) = (1 - Z_t) \exp\left\{ - \int_u^t \frac{Z_s}{1 - Z_s} \lambda_s ds \right\}$$

we obtain that, for $t > u$

$$g_t(u) = F_t(u) \frac{Z_u}{1 - Z_u} \lambda_u$$

and

$$dg_t(u) = \frac{Z_u}{1 - Z_u} \lambda_u dF_t(u) = g_t(u) \frac{1}{1 - Z_t} dM_t$$

Construction of supermartingales valued in $[0, 1]$

- **Starting from dual projections:** If K is an $\mathbb{F}$-predictable increasing process such that $G_t := \mathbb{E}[K_\infty - K_t | \mathcal{F}_t] \leq 1$, then G is a supermartingale valued in $[0, 1]$ and is the Azéma supermartingale of a random time, and K will be the dual predictable projection of A.
- **Starting from dual projections:** If K is an $\mathbb{F}$-optional increasing process such that $\widetilde{G}_t := \mathbb{E}[K_\infty - K_{t-} | \mathcal{F}_t] \leq 1$, then $\widetilde{G}$ is a supermartingale valued in $[0, 1]$ and K will be the dual optional projection of A

- Starting with intensity rate: Let B be a Brownian motion. The solution of

$$dG_t = -\lambda_t G_t dt + b_t G_t(1 - G_t)dB_t, \; G_0 = 1$$

where $\lambda \geq 0$ and b are two bounded optional processes is a supermartingale valued in $[0, 1]$ and λ is the intensity rate of the associated random times. The conditional density of τ is, for $t > u$

- For $t < u$, if λ is deterministic, and setting $\Lambda_t = \int_0^t \lambda(s)ds$ and $N_t = G_t e^{\Lambda(t)}$

$$\mathbb{E}[G_u|\mathcal{F}_t] = e^{-\Lambda(u)}\mathbb{E}[N_u|\mathcal{F}_t] = N_t e^{-\Lambda(u)}$$

so that $g_t(u) = \lambda(u)e^{-\Lambda(u)}N_t$.
If follows that

$$g_t(u) = \lambda(u)e^{-\Lambda(u)}N_t\mathbf{1}_{\{t \leq u\}} + \lambda(u)e^{-\Lambda(u)}\frac{N_u}{\nu_u}\nu_t\mathbf{1}_{\{t > u\}}$$

where $d\nu_t = -b_t e^{-\Lambda(t)}\nu_t N_t dB_t$.

- For Γ a continuous increasing $\mathbb{F}$ adapted process

$$dG_t = -G_t d\Gamma_t + b_t G_t(1 - G_t) dB_t, \; G_0 = 1$$

is a supermartingale valued in $[0, 1]$. This is in particular the case for last passage times, where Γ involves a local time, or when $G = Y \wedge 1$ for a positive supermartingale Y.

References: Generalities

- Aksamit, A and Jeanblanc, M.(2017) *Enlargement of Filtration with Finance in View.* SpringerBriefs in Quantitative Finance, Springer, (2017).
- Jeulin, T. (1980). *Semi-martingales et grossissement d'une filtration. Lecture Notes in Math. 833,* Springer, Berlin.
- Mertens, J.-F. (1972). Théorie des processus stochastiques généraux : applications aux surmartingales, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, **22**, 45-72.

References: Construction of default times

- Bélanger, A. and Shreve, S.E. and Wong, D., A unified model for credit derivative", Math. Finance, 14, 317-350, 2004
- El Karoui, N. and Jeanblanc, M. and Jiao, Y., What happens after a default: the conditional density approach, Stochastic Processes and their Appl., 2010, 120, 7, 1011-1032
- Gapeev, P., Jeanblanc, M., Li, L. and Rutkowski, M. (2010) Constructing random times with given survival processes and applications to valuation of credit derivatives. In: *Contemporary Quantitative Finance*, Springer, pp. 255–280.
- Jeanblanc, M. and Song, S., Explicit Model of Default Time with given Survival Probability, *SPA*, 121, 1678-170, 2011
- Jeanblanc, M. and Song, S. (2011). Random times with given survival probability and their $\mathbb{F}$-martingale decomposition formula. *SPA* **121** 1389–1410.
- Jeanblanc, M. and Li, L. (2019) Modeling of Default Times,

Other related references

- Jiao, Y. and Li, S. (2015) Generalized density approach in progressive enlargement of filtrations, Electronic Journal of Probability, 20
- Gehmlich, F. and Schmidt, Th. (2016) Dynamic Defaultable Term Structure Modeling Beyond the Intensity Paradigm, Mathematical Finance,**28**, 1, 211-239
- Kardaras, C.: *On the stochastic behaviour of optional processes up to random times*, Annals of Applied Probability, 25, 2 (2015), pp. 429–464.

ihp Institut Henri Poincaré

Émilie Du Châtelet

ihp Institut Henri Poincaré

Sofia Kovalevskaïa

©Library of Congress

ihp Institut Henri Poincaré

Emmy Noether

ihp Institut Henri Poincaré

Sophie Germain

ihp Institut Henri Poincaré

Ada Lovelace


SI ON AVAIT SON ANNIVERSAIRE DEUX FOIS PAR AN
ON VIVRAIT DEUX FOIS PLUS VIEUX
© casterman - geluck

Thank you for your attention