Longstaff Schwartz algorithm and Neural Network regression

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Conférence in honor of Nicole El Karoui
**INTRODUCTION**

- *Ongoing work with Jérôme Lelong, ENSIMAG*

- Computing an American option involving a large number of assets remains a numerically difficult task

- A hope: Neural Network (NN) can (may) help to reduce the computational burden.

- Some previous works using NN for optimal stopping (not LS algorithm though)
  
  

- What can be proved for a LS algorithm using NN approximation?
Computing Bermudan options prices

- A discrete time (discounted) payoff process \((Z_{T_k})_{0 \leq k \leq N}\) adapted to \((\mathcal{F}_{T_k})_{0 \leq k \leq N}\). \(\max_{0 \leq k \leq N} |Z_{T_k}| \in L^2\).
- The (discounted) value of the Bermudan option at times \((T_k)_{0 \leq k \leq N}\) is given by
  \[
  U_0 = \sup_{\tau} \mathbb{E}(Z_\tau).
  \]
- Standard dynamic programming algorithm (→ “Tsitsiklis-Van Roy” type algorithms).
  \[
  \begin{cases}
  U_{T_N} = Z_{T_N} \\
  U_{T_k} = \max (Z_{T_k}, \mathbb{E}[U_{T_{k+1}} | \mathcal{F}_{T_k}])
  \end{cases}
  \]
- Using optimal stopping times \(\tau_k\) at time \(k\) rewrites as (→ “Longstaff-Schwartz” algorithm)
  \[
  \begin{cases}
  \tau_N = T_N, \quad \text{then for } 0 < k < N \\
  \tau_k = T_k \mathbf{1}_{\{Z_{T_k} \geq \mathbb{E}[Z_{\tau_{k+1}} | \mathcal{F}_{T_k}]\}} + \tau_{k+1} \mathbf{1}_{\{Z_{T_k} < \mathbb{E}[Z_{\tau_{k+1}} | \mathcal{F}_{T_k}]\}}, \\
  U_0 = \max(Z_0, \mathbb{E}[Z_{\tau_1}]).
  \end{cases}
  \]
THE MARKOVIAN CONTEXT

- Markovian context: \((X_t)_{0 \leq t \leq T}\) is a Markov process and 
  \(Z_{T_k} = \phi_k(X_{T_k})\).

\[
\mathbb{E}[Z_{\tau_k+1} | \mathcal{F}_{T_k}] = \mathbb{E}[Z_{\tau_k+1} | X_{T_k}] = \psi_k(X_{T_k})
\]

- Because of \(L^2\) assumption, \(\psi_k\) can be computed by a minimization problem

\[
\inf_{\psi \in L^2(\mathcal{L}(X_{T_k}))} \mathbb{E} \left[ |Z_{\tau_k+1} - \psi(X_{T_k})|^2 \right]
\]

- LS = using equation (2), if you have an approximation of \(\tau_{k+1}\) using simulation, you can obtain by minimization an approximation of \(\mathbb{E}[Z_{\tau_k+1} | X_{T_k}]\) and so obtain an approximation of \(\tau_k\).
Different numerical strategies

- Longstaff/Schwartz type algorithms rely on direct approximation of stopping times and use of the same simulated paths for all time steps (obvious and large computational gains).
- The standard numerical (LS) approach: approximate the space $L^2$ by a finite dimensional vector space (polynomials, ...)
- We investigate the use of Neural Networks to approximate $\psi_k$.
- Kohler et al. [2010]: neural networks but in a different context (approximation of the value function Tsitsiklis and Roy [2001], equation (1)) and re-simulation of the paths at each time steps.
\section*{LS: truncation step}

- $(g_k, k \geq 1)$ is an $L^2(L(X))$ basis and $\Phi_p(X, \theta) = \sum_{k=1}^{p} \theta_k g_k(X)$.
- Backward approximation of iteration policy using (2),
  \[
  \begin{aligned}
  \hat{\tau}_N^p &= T_N \\
  \hat{\tau}_n^p &= T_n \mathbf{1}\{Z_{T_n} \geq \Phi_p(X_{T_n}; \hat{\theta}_n^p)\} + \hat{\tau}_{n+1}^p \mathbf{1}\{Z_{T_n} < \Phi_p(X_{T_n}; \hat{\theta}_n^p)\}
  \end{aligned}
  \]
- with conditional expectation computed using a Monte Carlo minimization problem: $\hat{\theta}_n^p$ is a minimizer of
  \[
  \inf_{\theta} \mathbb{E}\left(\left| \Phi_p(X_{T_n}; \theta) - Z_{\hat{\tau}_{n+1}^p} \right|^2\right).
  \]
- Price approximation: $U_0^p = \max\left(Z_0, \mathbb{E}\left(Z_{\hat{\tau}_1^p}\right)\right)$. 
The LS algorithm

- \((g_k, k \geq 1)\) is an \(L^2(\mathcal{L}(X))\) basis and \(\Phi_p(X, \theta) = \sum_{k=1}^p \theta_k g_k(X)\).
- Paths \(X_{T_0}^{(m)}, X_{T_1}^{(m)}, \ldots, X_{T_N}^{(m)}\) and payoff paths \(Z_{T_0}^{(m)}, Z_{T_1}^{(m)}, \ldots, Z_{T_N}^{(m)}\), \(m = 1, \ldots, M\).
- Backward approximation of iteration policy,

\[
\begin{align*}
\hat{\tau}_N^{p,(m)} &= T_N \\
\hat{\tau}_n^{p,(m)} &= T_n \mathbf{1} \left\{ Z_{T_n}^{(m)} \geq \Phi_p(X_{T_n}^{(m)}; \hat{\theta}_n^{p,M}) \right\} + \hat{\tau}_{n+1}^{p,(m)} \mathbf{1} \left\{ Z_{T_n}^{(m)} < \Phi_p(X_{T_n}^{(m)}; \hat{\theta}_n^{p,M}) \right\}
\end{align*}
\]

- with conditional expectation computed using a Monte Carlo minimization problem: \(\hat{\theta}_n^{p,M}\) is a minimizer of

\[
\inf_{\theta} \frac{1}{M} \sum_{m=1}^M \left| \Phi_p(X_{T_n}^{(m)}; \theta) - Z_{\tau_{n+1}^{p,(m)}}^{(m)} \right|^2.
\]

- Price approximation: \(U_0^{p,M} = \max \left( Z_0, \frac{1}{M} \sum_{m=1}^M Z_{\tau_1^{p,(m)}}^{(m)} \right) \).
REFERENCE PAPERS

- Description of the algorithm:
  

- Rigorous approach:
  
  
  - $U_0^p$ converge to $U_0$, $p \to +\infty$
  - $U_0^{p,M}$ converge to $U_0^p$, $M \to +\infty$ a.s.
  - “almost” a central limit theorem
**The modified algorithm**

- In LS algorithm replace approximation on a Hilbert basis $\Phi_p(., \theta)$ by a Neural Network.
- The optimization problem is non linear, non convex, ...
- Aim: extend the proof of (a.s.) convergence results
A quick view of Neural Networks

- In short a NN: \( x \rightarrow \Phi_p(x, \theta) \in \mathbb{R}, \theta \in \mathbb{R}^d, d \text{ large} \)
- \( \Phi_p = A_L \circ \sigma \circ A_{L-1} \circ \cdots \circ \sigma \circ A_1, L \geq 2, L \text{ fixed} \)
- \( A_l(x_l) = w_l x_l + \beta_l \) (affine functions)
- \( p \) “maximum number of neurons per layer” (i.e. sizes of the \( w_l \) matrix)
- \( \sigma \) a fixed non linear (called activation function) applied component wise
- \( \theta := \text{parameter} (w, \beta) \) of all the different layers
- Restriction to the compact set \( \Theta_p = \{ \theta : |\theta| \leq \gamma_p \} \) and assume \( \lim_{p \to \infty} \gamma_p = \infty \).
- \( NN_p = \{ \Phi_p(\cdot, \theta) : \theta \in \Theta_p \} \)
- \( NN_\infty = \bigcup_{p \in \mathbb{N}} NN_p \)
**Hypothesis H**

- there exist $\delta > 0$ and $q \geq 1$

\[
\begin{align*}
\left| \Phi_p(x, \theta_1) - \Phi(x, \theta_2) \right| &\leq \kappa |\theta_1 - \theta_2|^\delta (1 + |x|^q) \\
|\Phi_p(x, \theta_1)| &\leq \kappa_q (1 + |x|^q)
\end{align*}
\]

- $\mathbb{E}[|X_t|^{2q}] < \infty$ for all $0 \leq t \leq T$.
- For all $p, n < N$, $\mathbb{P} (Z_{T_n} = \Phi_p(X_{T_n}; \theta_{p}^n)) = 0$.
- the family of optimization problems admit a unique minimizer $\theta_{n}^p$

\[
\inf_{\theta \in \Theta_p} \mathbb{E} \left( \left| \Phi_p(X_{T_n}; \theta) - Z_{\hat{\tau}_{n+1}}^p \right|^2 \right).
\]
THE RESULT

Theorem

Under hypothesis $\mathbf{H}$

- **Convergence of the approximation in $p$**

$$\lim_{p \to \infty} \mathbb{E}[Z_{\tau_n}^p | \mathcal{F}_{T_n}] = \mathbb{E}[Z_{\tau_n} | \mathcal{F}_{T_n}] \text{ in } L^2(\Omega) \quad (i.e. \ U_0^p \to U_0).$$

- **SLLN : for every } k = 1, \ldots, N, \text{**

$$\lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} Z_{\tau_k}^{(m)} = \mathbb{E} \left[ Z_{\tau_k}^p \right] \quad a.s. \quad (i.e. \ U_0^{p,M} \to U_0^p)$$
Convergence of the approximation in $p$

A simple consequence of the Hornik theorem (Hornik [1991]).
- Also known as "Universal Approximation Theorem".

Theorem (Hornik)

Assume that the function $\sigma$ is non constant and bounded. Let $\mu$ denote a probability measure on $\mathbb{R}^r$, then $NN_{\infty}$ is dense in $L^2(\mathbb{R}^r, \mu)$.

- Corollary: If for every $p$, $\alpha_p \in \Theta_p$ is a minimizer of

$$\inf_{\theta \in \Theta_p} \mathbb{E}[|\Phi_p(X; \theta) - Y|^2],$$

$(\Phi_p(X; \alpha_p))_p$ converges to $\mathbb{E}[Y|X]$ in $L^2(\Omega)$ when $p \to \infty$.

- Proof of the convergence of the "non-linear approximation" $\Phi_p(X; \theta)$. 
Convergence of Monte-Carlo approximation

- $p$ fixed, $N \rightarrow +\infty$
- Minimisation problem is now non linear, need more abstract arguments to prove convergence
- Two ingredients ("old" results)
- First result: approximation of minimization problems

**Lemma (Rubinstein and Shapiro [1993])**

- $(f_n)_n$ defined on a compact set $K \subset \mathbb{R}^d$. $v_n = \inf_{x \in K} f_n(x)$
- $x_n$ a sequence of minimizers $f_n(x_n) = \inf_{x \in K} f_n(x)$.

If $(f_n)_n$ converges uniformly on $K$ to a continuous function $f$. $v^* = \inf_{x \in K} f(x)$, then $v_n \rightarrow v^*$.

Moreover if $x^*$ is the unique minimizer of $f$, then $x_n \rightarrow x^*$. 
Convergence of Monte-Carlo approximation

- Second result: SLLN in Banach spaces (Ledoux and Talagrand [1991], date back to Mourier [1953]).

**Lemma**

Let \((\xi_i)_{i \geq 1}\) i.i.d. \(\mathbb{R}^m\)-valued, \(h : \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}\). If

- a.s., \(\theta \in \mathbb{R}^d \mapsto h(\theta, \xi_1)\) is continuous,
- \(\forall K > 0, \mathbb{E} \left[ \sup_{|\theta| \leq K} |h(\theta, \xi_1)| \right] < +\infty\).

Then

\[
\lim_{n \to \infty} \sup_{|\theta| \leq K} \left| \frac{1}{n} \sum_{i=1}^{n} h(\theta, \xi_i) - \mathbb{E}[h(\theta, \xi_1)] \right| = 0 \text{ a.s.}
\]
Convergence of Monte-Carlo approximation

- Direct application of this results give convergence of
  \[ \hat{\theta}_N^p, M \to \theta_N^p \]

- Using backward iteration, following Clément et al. [2002], give, for \( n \leq N \)
  \[ \hat{\theta}_n^p, M \text{ converges to } \theta_n^p \text{ a.s. as } M \to \infty. \]

- Convergence of \( \hat{\theta}_n^p, M \) needed to use backward recurrence

- ...
Numerical aspects not covered here

- Numerical experiments yet to be done (work in progress)
- Numerical resolution of the optimization problem ...
- Stochastic gradient descent is used to obtain a (local?) minimum
- In principle less efficient than direct computation of the minimum (in the linear case)
- ...

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BIBLIOGRAPHIE I


