

Markov birthdays of the Hawkes process with general immigrants

Alexandre Boumezoued

23 mai 2019



Hawkes cluster representation

- Counting process with shot noise intensity

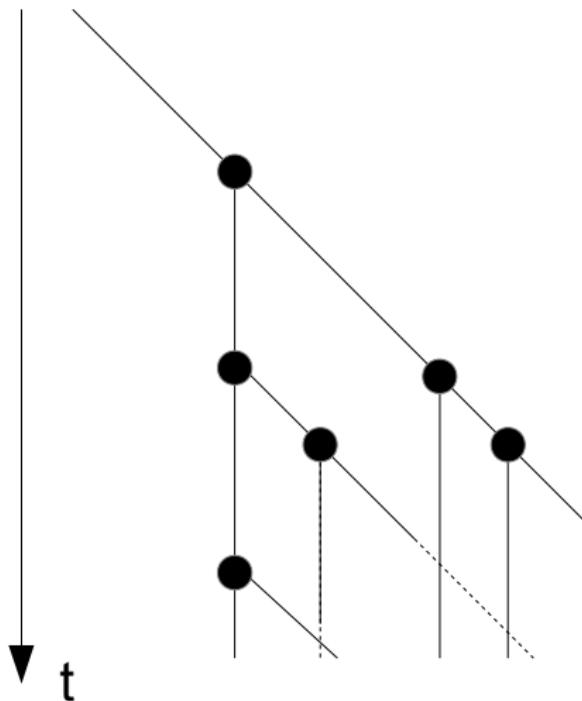
$$\lambda_t = \mu + \sum_{T_n < t} \phi(t - T_n)$$

- Popularity : **cluster representation** [Hawkes and Oakes(1974)]
- Immigrants arrive according to a Poisson process with intensity μ
- Each immigrant generates a **cluster of offsprings** with the rule : each point located at s generates further points according to a (non-homogenous) Poisson ($\phi(t - s)$)

⇒ Appealing property : **Clustering**

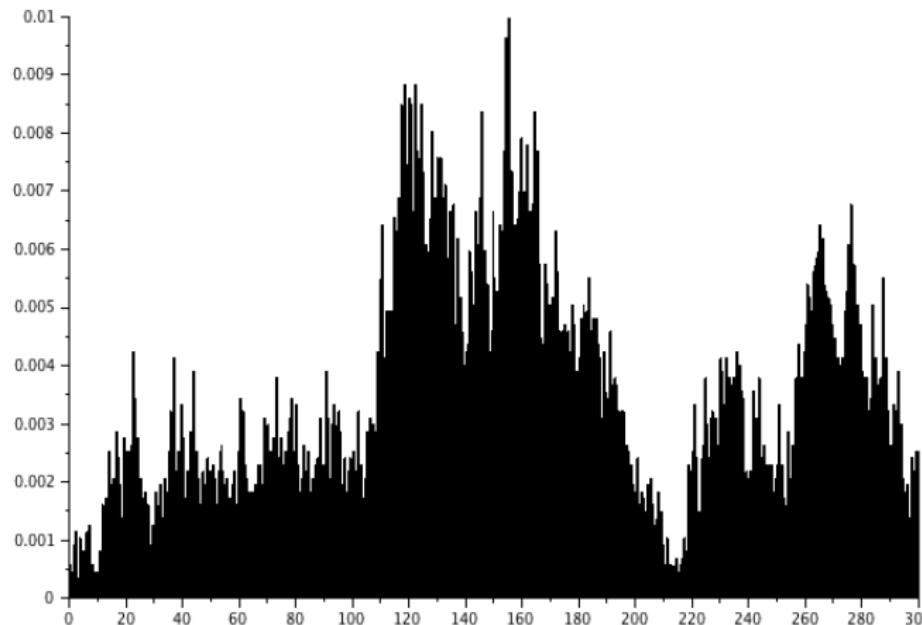


Hawkes cluster representation



Clustering

- Number of events per time unit (density)



Hawkes for applications

- neuroscience
- biology
- cosmology
- ecology
- epidemiology
- DNA modelling
- insurance : ruin theory, lapse risk, **cyber risk** (ongoing)
- finance : credit risk, contagion, market microstructure



Branching ratio

- Each indiv. at s gives birth with rate $t \mapsto \phi(t-s)$
- $\int_{\mathbb{R}_+} \phi(t) dt = \|\phi\| \equiv \text{mean number of children per individual}$
- Cluster representation **used in the stationary case** $\|\phi\| < 1$
- **Size of a cluster** of one immigrant :

$$\underbrace{\|\phi\|}_{\text{children}} + \underbrace{\|\phi\|^2}_{\text{grandchildren}} + \|\phi\|^3 + \dots = \frac{\|\phi\|}{1 - \|\phi\|}$$

Branching ratio \sim proportion of endogenous events

$$\frac{\frac{\|\phi\|}{1 - \|\phi\|}}{1 + \frac{\|\phi\|}{1 - \|\phi\|}} = \|\phi\|$$



Distribution properties

- **Cluster representation** of Hawkes and Oakes (1974) :
⇒ used for the computation of distribution properties **in the stationary case**
 - Adamopoulos (1975)
 - [Brémaud and Massoulié(2002)] [Bartlett spectrum]
 - Saichev and Sornette (2011)
 - Jovanovic et al. (2014)

In the **non-stationary case** : focus on $\phi(t) = \alpha e^{-\beta t}$
⇒ In this case, (λ_t) is a **Markov process** [Oakes(1975)]

Focus of this presentation : distribution properties of the (general linear) Hawkes process based on **general (exponential) kernels**

General (linear) Hawkes process

- Extended (linear) Hawkes process : (N_t) with intensity

$$\lambda_t = \mu(t) + \sum_{S_k < t} \Psi_t(t - S_k, Y_k) + \sum_{T_n < t} \Phi_t(t - T_n, X_n)$$

- T_n are the jumps of (N_t) (**self-excitation**)
- S_k form a NHPP with intensity $\rho(t)$ (**external excitation**)
- X_n (resp. Y_k) are iid with distribution G (resp. H)



General (linear) Hawkes process

- Extended (linear) Hawkes process : (N_t) with intensity

$$\lambda_t = \mu(t) + \sum_{S_k < t} \Psi_t(t - S_k, Y_k) + \sum_{T_n < t} \Phi_t(t - T_n, X_n)$$

- T_n are the jumps of (N_t) (**self-excitation**)
- S_k form a NHPP with intensity $\rho(t)$ (**external excitation**)
- X_n (resp. Y_k) are iid with distribution G (resp. H)
- Literature :

- Brémaud (2002) *Hawkes process with general immigrants*
- Dassios & Zhao (2011) *Dynamic contagion process*
- Filimonov & Sornette (2014)
Hawkes with renewal immigrants / EM Algorithm on simulated data
- Rambaldi *et al.* (2014) Application to FX market activity



Birth-immigration measure-valued process

- Each individual has a characteristic x and an age a
- Population of **external chocs** (1) :

$$Z_t^{(1)}(da, dx) = \sum_{S_k \leq t} \delta_{(t-S_k, Y_k)}(da, dx)$$

- **Hawkes population** (2) :

$$Z_t^{(2)}(da, dx) = \sum_{T_n \leq t} \delta_{(t-T_n, X_n)}(da, dx)$$

⇒ For each map $f(a, x)$, $\langle Z_t^{(2)}, f \rangle = \sum_{T_n \leq t} f(t - T_n, X_n)$

- The **Hawkes process** is the size of population (2) :

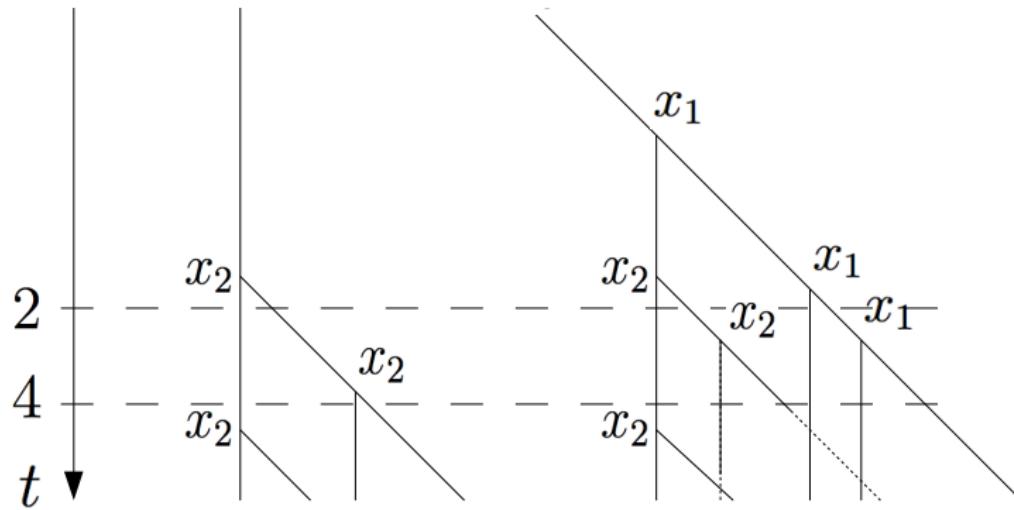
$$N_t = \langle Z_t^{(2)}, \mathbf{1} \rangle$$

- The **Hawkes intensity process** is

$$\lambda_t = \mu(t) + \langle Z_{t-}^{(1)}, \Psi_t \rangle + \langle Z_{t-}^{(2)}, \Phi_t \rangle$$



Dynamics of the two populations



Immigration-birth dynamics

- Recall : extended (linear) Hawkes process : (N_t) with intensity

$$\lambda_t = \mu(t) + \sum_{S_k < t} \Psi_t(t - S_k, Y_k) + \sum_{T_n < t} \Phi_t(t - T_n, X_n)$$

Interpretation :

- Immigrants arrive in population (1) at times S_k with rate $\rho(t)$
- Other immigrants come to population (2) with rate $\mu(t)$
- If an individual (a, x) lies in population (1) (resp. (2)) at time t , it gives birth with rate $\Psi_t(a, x)$ (resp. $\Phi_t(a, x)$) to individuals with age 0 and characteristics drawn with distribution H (resp. G).
- All individuals that are born belong to population (2).



Dynamics - special case

Assumption (example)

Standard Hawkes process such that $\phi^{(n)} = c_{-1} + \sum_{k=0}^{n-1} c_k \phi^{(k)}$

$$\langle Z_t, \phi \rangle = m_0 N_t + \int_0^t \langle Z_s, \phi' \rangle ds$$

⋮

$$\langle Z_t, \phi^{(k)} \rangle = m_k N_t + \int_0^t \langle Z_s, \phi^{(k+1)} \rangle ds$$

⋮

$$\langle Z_t, \phi^{(n-1)} \rangle = m_{n-1} N_t + \int_0^t \left\{ c_{-1} N_s + \sum_{k=0}^{n-1} c_k \langle Z_s, \phi^{(k)} \rangle \right\} ds$$



Assumptions

[Denote $f^{(k)} \equiv \frac{\partial^k f(x,a)}{\partial a^k}$]

Main Assumption : (exponential kernel generalized)

The kernels Φ and Ψ satisfy

$$\Phi_t(a,x) = v(t)\phi(a,x) \text{ and } \Psi_t(a,x) = w(t)\psi(a,x)$$

■ where

$$\phi^{(n)} = c_{-1} + \sum_{k=0}^{n-1} c_k \phi^{(k)} \text{ and } v^{(p)}(t) = d_{-1}(t) + \sum_{l=0}^{p-1} d_l(t) v^{(l)}(t),$$

with initial conditions $\phi^{(k)}(0,x) = \phi_0^{(k)}(x)$,

■ and

$$\psi^{(m)}(a,x) = r_{-1} + \sum_{k=0}^{m-1} r_k \psi^{(k)}(a,x) \text{ and } w^{(q)}(t) = k_{-1}(t) + \sum_{l=0}^{q-1} k_l(t) w^{(l)}(t),$$

with initial conditions $\psi^{(k)}(0,x) = \psi_0^{(k)}(x)$.



Dynamics

- Let us denote (with convention $f^{(-1)} \equiv \mathbf{1}$)

$$X_t^{(k,l)} := \langle Z_t^{(2)}, \partial_a^{(k)} \partial_t^{(l)} \Phi_t \rangle \text{ and } Y_t^{(k,l)} := \langle Z_t^{(1)}, \partial_a^{(k)} \partial_t^{(l)} \Psi_t \rangle$$

$$\Rightarrow N_t = X_t^{-1,-1}$$

- Construct the $(n+1)(p+1)$ -dimensional process

$$M_t^{(2)} = \left(X_t^{(k,l)} \right)_{-1 \leq k \leq n-1, -1 \leq l \leq p-1}$$

- In the same way, a $(m+1)(q+1)$ dimensional process

$$M_t^{(1)} = \left(Y_t^{(k,l)} \right)_{-1 \leq k \leq m-1, -1 \leq l \leq q-1}$$



Dynamics

Lemma

$$dM_t^{(i)} = \int_{\mathbb{R}_+} W^{(i)}(t, x) N^{(i)}(dt, dx) + \left(C^{(i)} M_t^{(i)} + M_t^{(i)} D_t^{(i)} \right) dt$$

where e.g. for $i=1$,

- $W_{k,l}^{(1)} = w^{l-2}(t) \psi_0^{(k-2)}(x)$ for $1 \leq k \leq m+1$ and $1 \leq l \leq q+1$,
- $C_{i,i+1}^{(1)} = 1$ for $2 \leq i \leq m$ and $C_{m+1,j}^{(1)} = r_{j-2}$ for $1 \leq j \leq m+1$,
- $D_{i,i+1}^{(1)}(t) = 1$ for $2 \leq i \leq q$ and $D_{q+1,j}^{(1)}(t) = k_{j-2}(t)$ for $1 \leq j \leq q+1$.



Example : seasonal Hawkes process

- Hawkes model with intensity $\mu + \sum_{T_n < t} \phi_t(t - T_n)$
- with fertility function $\phi_t(a) = \cos^2(\alpha t) e^{-ca}$.

Mean intensity in a seasonal Hawkes model

Denote $u(t) = \mathbb{E}[\lambda_t]$. Then u is the solution of the following equation system

$$u'(t) = [\cos^2(\alpha t) - c] u(t) + y(t) + c\mu$$

$$y'(t) = -2\alpha [\sin(\alpha t) \cos(\alpha t) + 2\alpha] u(t) - cy(t) + 2\alpha^2 z(t) + 4\mu\alpha^2$$

$$z'(t) = u(t) - cz(t)$$



Example : variance for $n=2$

- Example for $n=2$: variance of the intensity in the critical case

Corollary

For the Hawkes process with $\phi(a) = \beta^2 a e^{-\beta a}$, $\beta > 0$, we have

$$\text{Var}(\lambda_t) = \beta \mu \left(-\frac{7}{128} + \frac{3\beta}{32} t + \frac{\beta^2}{16} t^2 + \frac{1-\beta t}{8} e^{-2\beta t} - \frac{9}{128} e^{-4\beta t} \right).$$



Laplace transform

- Note that $\text{Tr}({}^t u.M_t) = \sum_{k,l} u_{k,l} M_t^{k,l}$

Proposition

The joint Laplace transform is given by

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\text{Tr} \left({}^t u.M_T^{(1)} + {}^t v.M_T^{(2)} \right) \right) \right] \\ &= \exp \left\{ \int_0^t \int_{\mathbb{R}_+} \mu(s) G(x) (e^{\text{Tr}({}^t A_s W^{(2)}(s,x))} - 1) ds dx \right. \\ & \quad \left. + \int_0^t \int_{\mathbb{R}_+} \rho(s) H(x) (e^{\text{Tr}({}^t B_s W^{(1)}(s,x))} - 1) ds dx \right\} \end{aligned}$$

where B depends on A , which solves, with $A_T = v$,

$${}^t A'_s + {}^t A_s \cdot C^{(2)} + D_s^{(2)} \cdot {}^t A_s + \mathbf{1}_{i=j=2} \int_{\mathbb{R}_+} \left(e^{\text{Tr}({}^t A_s \cdot W^{(2)}(s,x))} - 1 \right) G(x) dx = 0$$



Numerical examples

u	0.1	0.3	0.5	0.7	0.9
Case 1, $\phi(a) = e^{-a}$	0.490	0.532	0.588	0.672	0.828
Case 2, $\phi(a) = ae^{-a}$	0.494	0.546	0.615	0.714	0.874

TABLE – Computed values of $\mathbb{E}[u^{N_T}]$ with $\mu = 0.15$ and $T = 5$.

k	0	1	2	3	4
Case 1, $\phi(a) = e^{-a}$	0.472	0.165	0.0894	0.0577	0.0407
Case 2, $\phi(a) = ae^{-a}$	0.472	0.203	0.113	0.0700	0.0451

TABLE – Computed values of $\mathbb{P}(N_T = k)$ with $\mu = 0.15$ and $T = 5$.



B. 2015b.

Population viewpoint on Hawkes processes.

([hal-01149752](#)) *Advances in Applied Probability* 48.2 (June 2016) .



Brémaud, P, L Massoulié. 2002.

Power spectra of general shot noises and hawkes point processes with a random excitation.

Advances in Applied Probability 205–222.



Dassios, Angelos, Hongbiao Zhao. 2011.

A dynamic contagion process.

Advances in applied probability 43(3) 814–846.



Hawkes, Alan G. 1971.

Spectra of some self-exciting and mutually exciting point processes.

Biometrika 58(1) 83–90.



Hawkes, Alan G, David Oakes. 1974.

A cluster process representation of a self-exciting process.

Journal of Applied Probability 493–503.



Oakes, David. 1975.

The markovian self-exciting process.

Journal of Applied Probability 69–77.

