Central limit theorem for discretization errors based on general stopping time sampling

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Joint work with N. Landon and U. Stazhynski.

1 A few examples of collaboration with Nicole

1.1 "Deep teaching"

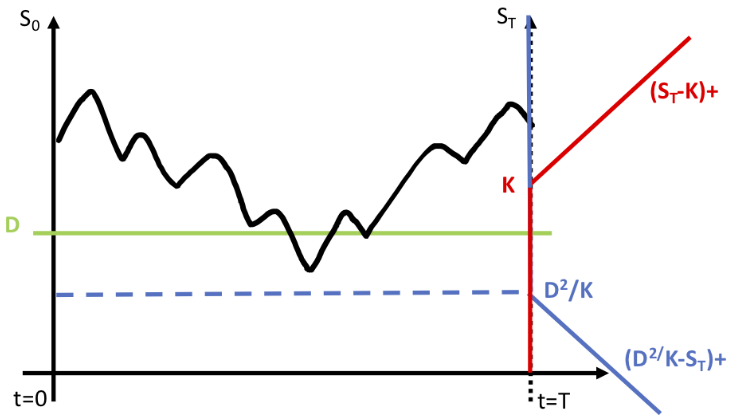
Pricing barrier options in Nicole's way

✓ Zero interest rate/zero dividend, Black-Scholes model, maturity T

✓ Down In Call regular ($D \leq K$), payoff $\mathbf{1}_{\min_t S_t \leq D}(S_T - K)_+$

Pricing/hedging:

1. If $S \leq D$,
   $\mathrm{DIC}(S, K, D) = \mathrm{Call}(S, K)$.
2. Si $S > D$,
   $\mathrm{DIC}(S, K, D) = \frac{K}{D}\mathrm{Put}\left(S, \frac{D^2}{K}\right)$
   $= \frac{K}{D}\mathrm{Call}\left(\frac{D^2}{K}, S\right) = \mathrm{Call}\left(D, \frac{KS}{D}\right)$.



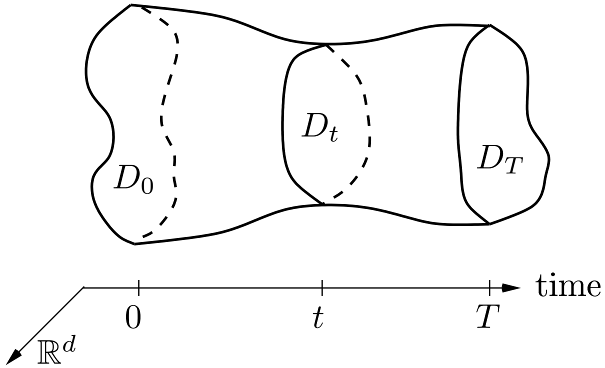
➠ Semi-static hedging

➠ Extension to drifted S using the power asset S_t^p (for suitable p)

➠ Direct proof for explicit distribution of $(S_T, \min_t S_t)$

1.2 Sensitivities with respect to boundaries

With C. Costantini (AMO 2006).



- ✓ X: SDE, $\tau = \inf\{s > t : (s, X_s^{t,x}) \notin D\}$.
- ✓ $u^D(t,x) = \mathbb{E}_{t,x}\Big(g(\tau, X_\tau) e^{-\int_t^\tau c(r,X_r)dr} - \int_t^\tau e^{-\int_t^s c(r,X_r)dr} f(s, X_s) ds \Big)$
- ✓ $D(\varepsilon) = \{(t,x) : (t, x + \epsilon\Theta(t,x)) \in D\}$, for $\epsilon \to 0$.

Theorem. Let $(t,x) \in D$ and set $\tau_\epsilon := \inf\{s > t : (s, X_s^{t,x}) \notin D(\varepsilon)\}$. Then the application $\varepsilon \mapsto u^{D(\varepsilon)}(t,x)$ is differentiable w.r.t. $\epsilon = 0$ and

$$\partial_\epsilon u^{D(\varepsilon)}(t,x)\Big|_{\epsilon=0} = \mathbb{E}_{t,x}\left[e^{-\int_t^\tau c(r,X_r)dr} [(\nabla u - \nabla g)\Theta](\tau, X_\tau) \right].$$

Link with the smooth-pasting property for American options.

2 Back to the CLT paper

2.1 Model

1. S: $\mathbb{R}^d$-valued Brownian semimartingale
2. Discretization of S at random stopping times $\tau_0^n = 0 < \tau_1^n < \cdots < \tau_{N_T^n}^n = T$
3. Random number of discretization times N_T^n
4. Discretization error $\mathcal{E}_t^n := \mathcal{E}_t^{n,1} + \mathcal{E}_t^{n,2}$ of the form

$$\mathcal{E}_t^{n,1} := \sum_{\tau_{i-1}^n < t} \int_{\tau_{i-1}^n}^{\tau_i^n \wedge t} \mathcal{M}_{\tau_{i-1}^n}(S_s - S_{\tau_{i-1}^n})\mathrm{d}s, \quad \mathcal{E}_t^{n,2} := \sum_{\tau_{i-1}^n < t} \int_{\tau_{i-1}^n}^{\tau_i^n \wedge t} (S_s - S_{\tau_{i-1}^n})^{\mathsf{T}} \mathcal{A}_{\tau_{i-1}^n} \mathrm{d}B_s.$$

Everything depends on $\varepsilon_n \to 0$.

Functional Central Limit Theorem (CLT) for the renormalized discretization error process $(\sqrt{N_t^n}\mathcal{E}_t^n)_{0\le t\le T}$?

2.2 Applications

1. *Integrated variance estimation [RR12][LZZ13][LMR⁺14].* Here $\mathrm{d}S_t = b_t \mathrm{d}t + \sigma_t \mathrm{d}B_t$ and the goal is to estimate $\int_0^t \mathrm{Tr}(\sigma_s \sigma_s^\mathsf{T})\mathrm{d}s$.

   Estimation error

$$\sum_{\tau_{i-1}^n < t} |\Delta S_{\tau_i^n \wedge t}|^2 - \int_0^t \mathrm{Tr}(\sigma_s \sigma_s^\mathsf{T})\mathrm{d}s = 2\int_0^t \Delta S_s^\mathsf{T} \sigma_s \mathrm{d}B_s + 2\int_0^t b_s^\mathsf{T} \Delta S_s \mathrm{d}s.$$

2. *Optimal tracking strategies [Fuk11b][Fuk11a][GL14][GS18a].* Minimization of the tracking error of a continuous-times strategy:

$$\int_0^t v(s, S_s)\mathrm{d}S_s - \sum_{\tau_{i-1}^n < t} v(\tau_{i-1}^n, S_{\tau_{i-1}^n})\Delta S_{\tau_i^n \wedge s} \approx \sum_{\tau_{i-1}^n < t} \int_{\tau_{i-1}^n}^{\tau_i^n \wedge t} \nabla_S v(\tau_{i-1}^n, S_{\tau_{i-1}^n})\Delta S_s \mathrm{d}S_s.$$

3. *Parametric estimation for processes [GJ93][GS18b].* Estimation of α, β in the drift/diffusion coefficients of SDE via minimum contrast estimators.

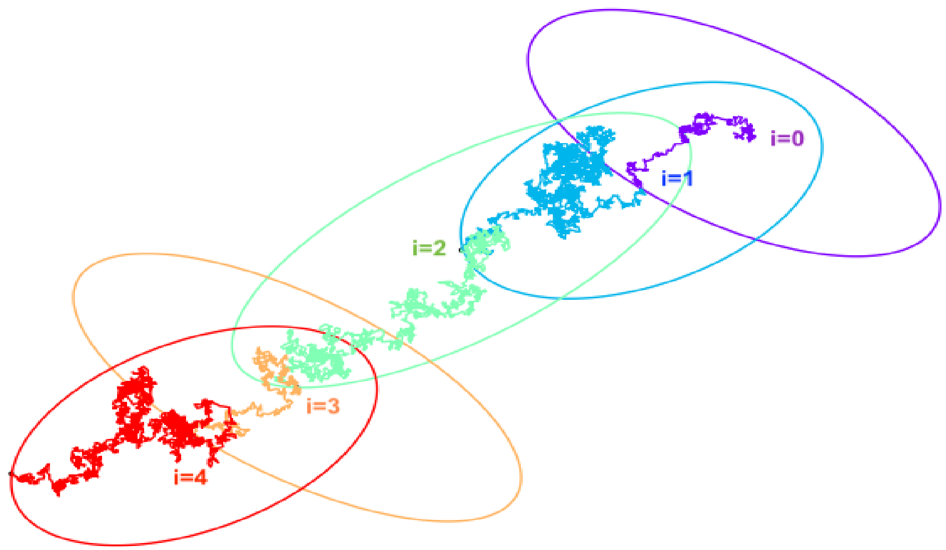
2.3 Form of the stopping time

✓ Quite general sequences of stopping times

✓ Combination of exit times by S of random domains and Poisson-like random times

$$\tau_i^n := \inf\{t > \tau_{i-1}^n : (S_t - S_{\tau_{i-1}^n}) \notin \varepsilon_n D^n_{\tau_{i-1}^n}\} \wedge (\tau_{i-1}^n + \varepsilon_n^2 G_{\tau_{i-1}^n}(U_{n,i}) + \Delta_{n,i}) \wedge T,$$

for some parameter $\varepsilon_n \to 0$, some stochastic domains $D^n_{.}$ indexed by time, some independent random variables $(U_{n,i})_{i,n}$, some negligible error terms $\Delta_{n,i}$.



2.4 Probabilistic model

✓ $(\Omega, \mathcal{F}, (\bar{\mathcal{F}}_t)_{0\le t\le T}, \mathbb{P})$ supporting a d-dimensional BM $(B_t)_{0\le t\le T}$

✓ right-continuous and $\mathbb{P}$-complete filtration

$(\mathbf{H}_S^{gen.})$: The process S on $[0,T]$ given by $\mathbf{S_t} = \mathbf{A_t} + \int_0^{\mathbf{t}} \sigma_{\mathbf{s}} \mathrm{d}\mathbf{B_s}$, where

✓ A η_A-Holder continous ($\eta_A \in (1/2, 1]$), with finite variation

✓ $(\sigma_t)_{0\le t\le T}$ $\eta_\sigma/2$-Holder continuous adapted invertible

$(\mathbf{H}_R)$:

1. Distance between 2 stopping times can not be large in expectation: For some adapted continuous non-decreasing process $(C_t^{(1)})_{0\le t\le T}$

$$\sup_{\tau_{i-1}^n < t \le T} \Big(\mathbb{E}_t(|S_{\tau_i^n} - S_{\tau_{i-1}^n}|^4) + |S_{t\wedge\tau_i^n} - S_{\tau_{i-1}^n}|^4 \Big) \le C_{\tau_{i-1}^n}^{(1)} \varepsilon_n^4. \tag{1}$$

2. Number of stopping times cannot be too large:

$$C_{(2)} := \sup_{n\ge 0} \left(\varepsilon_n^2 N_T^n\right) < +\infty, \quad \text{a.s.}. \tag{2}$$

$\mathcal{P}^\alpha$ = vector space spanned by α-homogeneous polynomial functions $\mathbb{R}^d \mapsto \mathbb{R}$

$(\mathbf{H}_{\mathcal{B}})$:

1. There is a linear operator $\mathcal{B}_{.}[.]$ from the vector space spanned by $\mathcal{P}^\alpha, \alpha = 2, 3, 4,$ into scalar adapted continuous process $(\mathcal{B}_t[f(\cdot)])_{0\leq t\leq T}$
2. $m_t := \dfrac{\mathcal{B}_t[f(x) := |x|^2]}{\mathrm{Tr}(\sigma_t\sigma_t^\top)} > 0$
3. $\exists\ g : [0,1] \to \mathbb{R}_+$ with $\lim_{\varepsilon\to 0}(g(\varepsilon) + \varepsilon^{2(1-\rho)}g(\varepsilon)^{-1}) = 0$ for some $\rho \in (0,1)$, such that for any $f \in \mathcal{P}^\alpha$ with $\alpha \in \{2,3,4\}$
$$\sup_{\tau_{i-1}^n < (T-g(\varepsilon_n))_+} \left| \varepsilon_n^{-\alpha}\mathbb{E}_{\tau_{i-1}^n}(f(S_{\tau_i^n} - S_{\tau_{i-1}^n})) - \mathcal{B}_{\tau_{i-1}^n}[f(\cdot)]\right| \leq C_{(3)}\varepsilon_n^\rho \quad \text{a.s.} \quad (3)$$
4. $\varepsilon_n^{-2}\#\{\tau_i^n : (T-g(\varepsilon_n))_+ \leq \tau_i^n \leq T\} \xrightarrow[n\to+\infty]{\text{a.s.}} 0.$

2.5 Main result

Theorem. Under these assumptions, there are some (explicit) random process Q and $\mathcal{K}$ (symmetric) and an m-dimensional Brownian motion W defined on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and independent of $\bar{\mathcal{F}}_T$ such that the following convergences hold:

1. the functional $\bar{\mathcal{F}}$-stable convergence in distribution

$$\varepsilon_{\mathbf{n}}^{-\mathbf{1}} \mathcal{E}_{\mathbf{t}}^{\mathbf{n}} \underset{[\mathbf{0},\mathbf{T}]}{\overset{\mathbf{d}}{\Longrightarrow}} \left(\int_{\mathbf{0}}^{\mathbf{t}} \mathcal{M}_{\mathbf{s}} \mathbf{Q}_{\mathbf{s}} \mathrm{d}\mathbf{s} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{Q}_{\mathbf{s}}^{\mathsf{T}} \mathcal{A}_{\mathbf{s}} \mathrm{d}\mathbf{B}_{\mathbf{s}} + \int_{\mathbf{0}}^{\mathbf{t}} \mathcal{K}_{\mathbf{s}}^{\mathbf{1/2}} \mathrm{d}\mathbf{W}_{\mathbf{s}} \right); \tag{4}$$

2. the uniform convergence in probability

$$\varepsilon_{\mathbf{n}}^{\mathbf{2}} \mathbf{N}_{\mathbf{t}}^{\mathbf{n}} \underset{\mathbf{n}\to+\infty}{\overset{\mathbf{u.c.p.}}{\longrightarrow}} \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{m}_{\mathbf{s}}^{-\mathbf{1}} \mathrm{d}\mathbf{s}. \tag{5}$$

Possible bias at the limit (specific to discretization on random times)

2.6 The case of stopping times of the form

$\tau_{\mathbf{i}}^{\mathbf{n}} := \inf\{\mathbf{t} > \tau_{\mathbf{i-1}}^{\mathbf{n}} : (\mathbf{S_t} - \mathbf{S}_{\tau_{\mathbf{i-1}}^{\mathbf{n}}}) \notin \varepsilon_{\mathbf{n}} \mathbf{D}^{\mathbf{n}}_{\tau_{\mathbf{i-1}}^{\mathbf{n}}}\} \wedge (\tau_{\mathbf{i-1}}^{\mathbf{n}} + \varepsilon_{\mathbf{n}}^{\mathbf{2}} \mathbf{G}_{\tau_{\mathbf{i-1}}^{\mathbf{n}}}(\mathbf{U_{n,i}}) + \boldsymbol{\Delta}_{\mathbf{n,i}}) \wedge \mathbf{T}.$

Domains assumptions:

✓ D^n: intersection of smooth C^2 domains sandwiched between balls (with stochastic radius : ω-ise compact). Includes stochastic polyhedrons.

✓

$$\sup_{n\geq 0} \left(\varepsilon_n^{-\eta_{\mathcal{D}}} \sup_{0\leq t\leq T} \mathtt{dist}(D_t^n, D_t) \right) < +\infty.$$

✓ Appropriate measurability conditions on G and

$$\mathbb{E}\left(|\Delta_{n,i}| \mid \bar{\mathcal{F}}_{\tau_{i-1}^n} \right) \leq p_{\tau_{i-1}^n} \varepsilon_n^{2+\eta}.$$

Theorem. The CLT holds with the process/opeator m and $\mathcal{B}$ defined as follows:

✓ $\tau(t) := \inf\{s \geq 0 : \sigma_t \tilde{W}_s \notin D_t\} \wedge G_t(U)$

✓

$$\mathcal{B}_t[f(\cdot)] := \tilde{\mathbb{E}}_t \left(f(\sigma_t \tilde{W}_{\tau(t)}) \right),\ t \in [0, T],$$

$$m_t := \tilde{\mathbb{E}}_t(\tau(t)),\ t \in [0, T],$$

with $U \sim \mathcal{U}(0, 1)$ be independent of $\tilde{W}$, both independent of $\bar{\mathcal{F}}_T$.

2.7 Explicit computations in dimension 1

Consider:

- ✓ $G_t(\cdot) \equiv +\infty$
- ✓ $D_t := (-\alpha_t, \beta_t) \subset \mathbb{R}$ for some adapted continuous a.s. positive processes α, β.

We obtain

$$m_t = \alpha_t \beta_t \sigma_t^{-2}, \qquad Q_t = \frac{1}{3}(\beta_t - \alpha_t), \qquad \mathcal{K}_t = \frac{(\mathcal{A}_t)^2}{18}(\alpha_t^2 + \beta_t^2 + \alpha_t\beta_t).$$

So finally we get

$$\sqrt{N_t^n}\mathcal{E}_t^n \xrightarrow[{[0,T]}]{d} \frac{1}{3}\sqrt{\int_0^t \frac{\sigma_s^2}{\alpha_s\beta_s}\mathrm{d}s}\Big(\int_0^t \mathcal{M}_s(\beta_s - \alpha_s)\mathrm{d}s + \int_0^t (\beta_s - \alpha_s)\mathcal{A}_s \mathrm{d}B_s$$
$$+ \frac{1}{\sqrt{2}}\int_0^t \mathcal{A}_s\sqrt{\alpha_s^2 + \beta_s^2 + \alpha_s\beta_s}\mathrm{d}W_s\Big).$$

We retrieve the one-dimensional result of [Fuk10, Theorem 3.1]. More general situations in [GLS18].

2.8 Ingredients of the proof

✓ Application of CLT from Jacod-Shirayev, convergence in probability...

✓ We switch to a.s. convergence: $\mathbb{R} \ni \mathcal{X}_n \underset{n\to+\infty}{\overset{\mathbb{P}}{\to}} \mathcal{X} \Longleftrightarrow$ for any subsequence $(\mathcal{X}_{\iota(n)})_{n\geq 0}$, $\exists \iota'$ s.t. $\mathcal{X}_{\iota\circ\iota'(n)} \underset{n\to+\infty}{\overset{a.s.}{\to}} \mathcal{X}$. ➠ we can assume $\sum_n \varepsilon_n^2 < +\infty$ and prove a.s. convergence

✓ a.s. convergence of martingales [GL14], $p > 0$

$$\sum_{\mathbf{n}} \sup_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]} |\mathbf{M_t}|^{\mathbf{p}} < +\infty \text{a.s.} \Longleftrightarrow \sum_{\mathbf{n}} \sup_{\mathbf{t}\in[\mathbf{0},\mathbf{T}]} \langle \mathbf{M_t}\rangle^{\mathbf{p/2}} < +\infty \text{a.s.}$$

➠ $$\varepsilon_n^2 \sum_{\tau_{i-1}^n < t} H_{\tau_{i-1}^n} \underset{n\to+\infty}{\overset{u.c.a.s.}{\longrightarrow}} \int_0^t H_s m_s^{-1} \mathrm{d}s,$$

$$\varepsilon_n^{2-\alpha} \sum_{\tau_{i-1}^n < t} H_{\tau_{i-1}^n} f_{\tau_{i-1}^n}(S_{\tau_i^n \wedge t} - S_{\tau_{i-1}^n}) \underset{n\to+\infty}{\overset{u.c.a.s.}{\longrightarrow}} \int_0^t H_s m_s^{-1} \mathcal{B}_s[f_s(\cdot)] \mathrm{d}s.$$

✓ Proof of stability estimates for exit times/position for perturbed domains

✓ Making the CLT characteristics "explicit"

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