Existence and optimality conditions in stochastic mean-field optimal control

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SUMMARY

Introduction

- **Game theory** is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers.
- **Game theory** is mainly used in economics, political science, and psychology, as well as logic, computer science, biology etc...
- **Existence** of mixed-strategy equilibria in two-person zero-sum games has been proved by J**John Von Neumann**, using Brouwer fixed-point theorem (**Theory of Games and Economic Behavior**, J. Von Neuman and Oskar Morgenstern.)

Introduction

- For general games, the Nash equilibrium is a solution involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and *no player has anything to gain by changing only his or her own strategy.*
- J. F. Nash has proved existence of a stable equilibrium in the space of **mixed strategies**.
- J. F. Nash: **Nobel prize** in Economics (1994) and **Abel Prize** in Mathematics(2015), with **21 published papers!!!!**.

Introduction

Nash equilibria in classical differential games
Let $S_1, ..., S_N$ be compact metric spaces, $J_1, ..., J_N$ be continuous real valued functions on $\prod_{i=1}^{N} S_i$. We denote by $\mathcal{P}(S_i)$ the compact metric space of all Borel probability measures defined on S_i.

Definition. *A Nash equilibrium in mixed strategies is a N-tuple* $(\overline{\pi}_1, \overline{\pi}_2, ..., \overline{\pi}_N) \in \prod_{i=1}^{N} \mathcal{P}(S_i)$ *such that, for any* $i = 1, ..., N$:
$J_i(\overline{\pi}_1, \overline{\pi}_2, ..., \overline{\pi}_N) \leq J_i((\overline{\pi}_j)_{j \neq i}, \pi_i)$ for every $\pi_i \in \mathcal{P}(S_i)$,
where

$$J_i(\pi_1, \pi_2, ..., \pi_N) = \int_{S_1 \times S_2 \times ... \times S_N} J_i(s_1, s_2, ..., s_N) d\pi_1(s_1) d\pi_2(s_2) ... d\pi_N(s_N)$$

Introduction

Remark. *Note that last condition is equivalent to*

$$J_i(\overline{\pi}_1, \overline{\pi}_2, ..., \overline{\pi}_N) \leq J_i((\overline{\pi}_j)_{j \neq i}, s_i) \text{ for every } s_i \in S_i.$$

Theorem (Nash, 1950) *Under the above assumptions, there exists at least one equilibrium point in mixed strategies.*

Theorem (Symmetric games) *If the game is symmetric, then there is an equilibrium of the form $(\overline{\pi}, \overline{\pi}, ..., \overline{\pi})$, where $\overline{\pi} \in P(S)$ is a mixed strategy.*
The game is symmetric if $S_i = S$ and
$J_{\sigma(i)}(s_{\sigma(1)}, s_{\sigma(2)}, ..., s_{\sigma(N)}) = J(s_1, s_2, ..., s_n)$

Introduction

The N Player Game.

Consider a stochastic differential game with N players, each player controlling his own private state X_t^i

$$\begin{cases} dX_t^i = b(t, X_t^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}, u_t^i)dt + \sigma(t, X_t^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}, u_t^i)dW_t^i \\ X_0 = x. \end{cases}$$

$$J^i(u^i) = E\left(\int_0^T h(t, X_t^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X^i}, u^i)dt + g(X_T^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X_T^i})\right)$$

$\left(u^1, u^2, ..., u^N\right)$ is a Nash equilibrium if $\forall 1 \leq i \leq N$, $\forall v \in \mathbb{A}$,

$$J^i(u^1, u^2, .., u^i .., u^N) \leq J^i(u^1, u^2, .., v .., u^N)$$

Introduction

Difficulties:

- Solve the HJB equation for large games.
- Numerical computations due to the dimension of the system.

When the number of players tends to infinity, can we expect some form of **averaging**?

The answer is given by the MEAN-FIELD GAME (MFG) THEORY invented by PL. Lions and J. M. Lasry

The MFG theory is to search for **approximate Nash equilibriums** in the case of small players.

Introduction

Remarks

- By small player, we mean a player who has very little influence on the overall system.
- This theory has been recently developed by J.-M. Lasry and P.-L. Lions in a series of papers (2006) and presented through several lectures (vidéos) by P.-L. Lions at the **College de France** 2008-2009.
- Its name comes from the analogy with the mean-field models in mathematical physics which analyzes the behavior of many identical particles (see Sznitman's lecture notes at Ecole d'été de Saint-Flour, Springer 1989).
- Related ideas have been developed independently, and at about the same time, by Huang-Caines-Malhamé (2006) in engineering.

Introduction

MFG solution can be resumed in the following steps

(i) Fix a deterministic function $\mu_t \in \mathcal{P}(\mathbb{R}^d)$

(ii) Solve the standard stochastic control problem

$$\begin{cases} dX_t = b(t, X_t, \mu_t, u_t)dt + \sigma(t, X_t^i, \mu_t, u_t)dW_t \\ \inf\limits_{a \in \mathbb{A}} E\left(\int\limits_0^T h(t, X_t, \mu_t, u)dt + g(X_T, \mu_T) \right) \end{cases}$$

(iii) Determine μ_t so that $P_{X_t} = \mu_t$.

If the fixed-point optimal control identified is in feedback form, $u_t = \alpha(t; X_t; P_{X_t})$ for some function α, then if the players use this strategy $u_t^i = \alpha(t; X_t^i; P_{X_t})$, then $\left(u_t^1, u_t^2, ..., u_t^N\right)$ should form an approximate Nash equilibrium.

Introduction

The solution of this problem is resumed in a coupled system of PDEs:

$$\begin{cases} \partial_t v(t;x) + \frac{\sigma^2}{2}\Delta_x v(t;x) + H(t,x,\mu_t,\nabla_x v(t,x),\alpha(t,x,\mu_t,\nabla_x v(t,x)) = 0 \\ \partial_t \mu_t - \frac{\sigma^2}{2}\Delta_x \mu_t + div_x\left(b(t,x,\mu_t,\nabla_x v(t,x),\alpha(t,x,\mu_t,\nabla_x v(t,x))\mu_t\right) = 0 \\ v(T;.) = g(;\mu_T);\ \ \mu_{0=}\delta_{x_0} \end{cases}$$

Huang, M., Malhamé, R. P., Caines, P. E. (2006). *Large population stochastic dynamic games: closed-loop MCKean-Vlasov systems and the nash certainty equivalence principle.* Comm. in Inf. and Systems, **6**(3), 221–252.

J.M. Lasry, P.L. Lions, *Mean-field games*. Japan. J. Math., **2** (2007) 229–260.

Introduction

The probabilistic method is based on solving "**forward backward stochastic differential equations**" and allows one to handle non Markovian models.

Carmona, R., Delarue, F., Probabilistic theory of mean field games with applications. I. Mean field FBSDEs, control, and games. *Probability Theory and Stochastic Modelling,* **83**. *Springer, Cham, 2018.*

Optimal control of MFSDEs

Optimal control of a mean-field SDE (MFSDE)

$$\begin{cases} dX_t = b(t, X_t, E\left(\Psi(X_t)\right), u_t)dt + \sigma(t, X_t, E\left(\Phi(X_t)\right), u_t)dW_t \\ X_0 = x. \end{cases}$$

The coefficients depend on the state X_t and on its distribution via a quantity $E\left(\Psi(X_t)\right)$, called: **mean-field term**. Minimize

$$J(u) \quad = \quad E\left(\int_0^T h(t, X_t, E(\varphi(X_t)), u_t)dt + g(X_T, E(\lambda(X_T))\right)$$

over a set of admissible controls $\mathcal{U}_{ad}$.
A control $\widehat{u}$ is optimal if $J(\widehat{u}) = \inf\left\{J(u); u \in \mathcal{U}_{ad}\right\}$.

Optimal control of MFSDEs

The state equation (**MFSDE**) is obtained as a limit of systems of interacting particles.

Lemma

Let $(X_t^{i,n})$, $i = 1, ..., n$, *defined by*

$$dX_t^{i,n} = b(t, X_t^{i,n}, \frac{1}{n}\sum_{i=1}^{n}\psi(X_t^{i,n}), u_t)dt + \sigma(t, X_t^{i,n}, \frac{1}{n}\sum_{i=1}^{n}\Phi(X_t^{i,n}), u_t)dW_t^i$$

Then $\lim_{n\to+\infty} E\left(\left|X_t^{i,n} - X_t^i\right|^2\right) = 0$, *where* (X_t^i) *are independent and solutions of the same MFSDE.*

B. Jourdain, S. Méléard, W. Woyczynski, *Nonlinear SDEs driven by Lévy processes and related PDEs*. Alea **4**, 1–29 (2008).

A.S. Sznitman, *Topics in propagation of chaos*. In Ecole de Probabilités de Saint Flour, XIX-1989. LN 1464, Springer, Berlin (1989).

Optimal control of MFSDEs

Applications to various fields:

- Allocation of economic resources.
- Exploitation of exhaustible resources, such as oil.
- Finance with small investors.
- Movement of large populations.

Optimal control of MFSDEs

Example

"**Mean-Variance Portfolio Selection**"

Consider a financial market : S_t^1(risky asset) and a bond S_t^0 (bank account) :

$$\begin{cases} dS_t^0 = \rho_t S_t^0 dt \\ dS_t^1 = \alpha_t S_t^1 dt + \sigma_t S_t^1 dW_t \end{cases}$$

If u_t is the proportion invested in S_t^1, then the value of the portfolio satisfies :

$dX_t = (\rho_t X_t + (\alpha_t - \rho_t)u_t)dt + \sigma_t u_t dW_t, X_0 = x$

Minimize the cost functional:

$J(u) = \frac{\gamma}{2} Var(X_T) - E(X_T) = \frac{\gamma}{2}\left(E(X_T^2) - E(X_T)^2\right) - E(X_T).$

Existence of optimal controls

The state equation

$$\begin{cases} dX_t = b(t, X_t, E(\Psi(X_t)), u_t)dt + \sigma(t, X_t, E(\Phi(X_t), u_t)dW_t \\ X_0 = x. \end{cases}$$

and the cost functional:

$$J(u) = E\left[\int_0^T h(t, X_t, E(\varphi(X_t)), u_t)dt + g(X_T, E(\lambda(X_T)))\right]$$

Without additional convexity conditions existence of a optimal strict control is not guaranteed. The idea is then to use **relaxed controls** which are measure valued controls.

Existence of optimal controls

Let $\mathbb{V}$ be the set of product measures μ on $[0, T] \times \mathbb{A}$ whose projection on $[0, T]$ coincides with the Lebesgue measure dt. $\mathbb{V}$ is compact for the topology of weak convergence.

Definition

A relaxed control on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a random variable $\mu = dt.\mu_t(da)$ with values in $\mathbb{V}$, such that $\mu_t(da)$ is progressively measurable with respect to $(\mathcal{F}_t)$ and such that for each t, $1_{(0,t]}.\mu$ is $\mathcal{F}_t-$measurable.

Remark

The set $\mathcal{U}_{ad}$ of strict controls is embedded into the set of relaxed controls by identifying u_t with $dt\delta_{u_t}(da)$.

El Karoui,N., Nguyen, D.H., Jeanblanc-Picqué, M., Compactification methods in the control of degenerate diffusions: existence of an optimal control, *Stochastics, 20 (1987), No. 3, 169-219*.

Existence of optimal controls

It was proved in [El Karoui and Méléard] that the relaxed state process corresponding to a relaxed control must satisfy a MFSDE driven by a martingale measure instead of a Brownian motion.

$$\begin{cases} dX_t = \int_{\mathbb{A}} b(t, X_t, E\left(\Psi\left(X_t\right)\right) a)\, \mu_t(da)dt \\ + \int_{\mathbb{A}} \sigma(t, X_t, E(\Phi(X_t), a)\, M(da, dt), \\ X_0 = x \end{cases}$$

where $M(dt, da)$ is continuous orthogonal martingale measure with intensity $\mu_t(da)dt$.

The relaxed cost is given:

$$J(\mu) = E\left(\int_0^T \int_A h(t, X_t, E(\varphi(X_t), a)\mu_t(da)dt + g(X_T, E(\lambda(X_T))\right)$$

N. El Karoui, S. Méléard, *Martingale measures and stochastic calculus*, Probab. Th. and Rel. Fields **84** (1990), no. 1, 83–101.

Existence of optimal controls

Theorem

Under $(\mathbf{H_1})$ *et* $(\mathbf{H_2})$*, an optimal relaxed control exists.*

Steps of the proof

- Let $(\mu^n)_{n\geq 0}$ a minimizing sequence, $\lim\limits_{n\to\infty} J(\mu^n) = \inf\limits_{q\in\mathcal{R}} J(\mu)$ and let X^n the state associated to μ^n.
- We prove that (μ^n, M^n, X^n) is tight.
- Using Skorokhod theorem, there exist a subsequence which converges strongly to $(\widehat{\mu}, \widehat{M}, \widehat{X})$, satisfying the state equation.
- Prove that $(J(\mu^n))_n$ converges to $J(\widehat{\mu}) = \inf\limits_{\mu\in\mathcal{R}} J(\mu)$ and conclude that $(\widehat{\mu}, \widehat{M}, \widehat{X})$ is optimal.

Existence of optimal controls

Corollary

Assume that
$P(t, X_t) = \left\{ \left(\widetilde{b}(t, X_t, E(\Psi(X_t), a) \right) ; \ a \in \mathbb{A} \right\} \subset \mathbb{R}^{d+d^2+1}$
is closed and convex, $\widetilde{b} = (b, \sigma\sigma^*, h)$. *Then the optimal relaxed control is realized as a strict control.*

The stochastic maximum principle

The state equation

$$\begin{cases} dX_t = \int_A b(t, X_t, E(X_t), a)\mu_t(da)dt + \int_A \sigma(t, X_t, E(X_t), a)M(dt, da) \\ X_0 = x, \end{cases}$$

The cost functional

$$J(\mu) = E\left[\int_0^T \int_A h(t, X_t, E(X_t), a)\mu_t(da)dt + g(X_T, E(X_T))\right].$$

The stochastic maximum principle

An optimal relaxed control exists.We derive necessary condtions for optimality in the form of Pontriagin maximum principle.
Let μ be an optimal relaxed control and X the optimal state.
The necessary conditions are given by

- two adjoint processes,
- a variational inequality.

The stochastic maximum principle

Assume

$(\mathbf{H_1})$

$$b : [0, T] \times \mathbb{R} \times \mathbb{R} \times A \longrightarrow \mathbb{R}$$
$$\sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times A \longrightarrow \mathbb{R}$$

are bounded, continuous, such that $b(t, ., ., a)$ and $\sigma(t, ., ., a)$ are C^2 in (x, y). Assume that the derivatives of order 1 and 2 are bounded continuous in (x, y, a).

$(\mathbf{H_2})$

$$h : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{A} \longrightarrow \mathbb{R}$$
$$g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

satisfy the same hypothesis as b and σ.

The stochastic maximum principle

Define the first and second order adjoint processes :

$$\begin{cases} dp(t) = & -\left[\overline{b}_x(t)p(t) + E\left(\overline{b}_y(t)p(t)\right) + \overline{\sigma}_x(t)q(t) + E\left(\overline{\sigma}_y(t)q(t)\right)\right. \\ & \left. -\overline{h}_x(t) - E\left(\overline{h}_y(t)\right)\right] dt + q(t)dW_t + dM_t \\ p(T) = & -\overline{g}_x(T) - E\left(\overline{g}_y(T)\right) \end{cases}$$

$$\begin{cases} dP(t) = & -\left[2\overline{b}_x(t)P(t) + \overline{\sigma}_x^2(t)P(t) + 2\overline{\sigma}_x(t)Q(t) + \overline{H}_{xx}(t)\right]dt \\ & +Q(t)dW_t + dN_t \\ P(T) = & -\overline{g}_{xx}(x(T)) \end{cases}$$

$\overline{f}(t) = f(t, X(t), \mu(t)) = \int_A f(t, X(t), a)\mu(t, da)$ and f stands for b_x, σ_x, h_x, b_y, σ_y, h_y, H_{xx}

The stochastic maximum principle

Denote the generalized Hamiltonian

$$\begin{gathered}\mathcal{H}^{(X(.),\mu(.))}(t, Y, E(Y), a) = \\ H(t, Y, E\left(Y\right), a, p(t), q(t) - P(t).\sigma(t, X_t, E(X_t), \mu(t))) \\ -\tfrac{1}{2}\sigma^2(t, Y, E\left(Y\right), a)P(t)\end{gathered}$$

where

$$\begin{gathered}H(t, X, E\left(X\right), a, p(t), q(t)) = \\ b(t, X, E(X), u).p + \sigma(t, X, E(X), u).q - h(t, X, E(X), u)\end{gathered}$$

is the usual Hamiltonian.

The stochastic maximum principle

Theorem

(The relaxed maximum principle)
Let (μ, X) *an optimal couple, then there exist* (p, q) *et* (P, Q)*, solutions of adjoint equations s.t*

$$E\int_0^T \mathcal{H}^{(X(t),\mu(t))}(t, X(t), \mu(t))dt = \sup_{a\in A} E\int_0^T \mathcal{H}^{(X(t),\mu(t))}(t, X(t), a)dt$$

The stochastic maximum principle

Idea of the proof

- Step1
- Approximate the optimal relaxed control $\mu_t(da)\,dt$ by a sequence (u_t^n), *such that* $\left(\delta_{u_t^n}(da)\,dt\right)$ *converges in* $\mathbb{V}$ *to* $\mu_t(da)\,dt,\ P-a.s.$
- (M^n) converges to M.
- $\lim\limits_{n\to\infty} E\left[\sup\limits_{0\leq t\leq T}\left|X_t^n - X_t\right|^2\right] = 0.$
- $\lim\limits_{n\to\infty} J(u^n) = J(\mu)$

The stochastic maximum principle

- Step 2: The controls u_t^n are nearly optimal. Apply Ekeland's variational principle and Buckdahn-Djehiche-Li maximum principle [3], to derive necessary conditions for near optimality.

$$E\left(\int_0^T \mathcal{H}^{(X^n(t),u^n(t))}(t,X^n(t),u^n(t))dt\right) \geq$$
$$\sup_{a\in A} E\left(\int_0^T \mathcal{H}^{(X^n(t),u^n(t))}(t,X^n(t),a)dt\right) - \varepsilon^{1/3}$$

- Etape3: Pass to the limit in the state equation, the adjoint equations and the Hamiltonians.

El Karoui, N.; Peng, S.; Quenez, M. C. Backward stochastic differential equations in finance. Math. Finance 7 (1997), no. 1, 1–71

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Thank you very much