Calcul d'Ito sans probabilités: pathwise calculus for non-anticipative functionals of irregular paths

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Föllmer, H. [**Föllmer, Hans**]

Calcul d'Itô sans probabilités. (French)

Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French), pp. 143–150, *Lecture Notes in Math.*, 850, *Springer, Berlin*, 1981.

L'auteur montre que le calcul d'Itô peut se faire trajectoire par trajectoire, à l'aide des sommes de Riemann par rapport à une classe de fonctions réelles à variation quadratique, $x(t)$, continues à droite et pourvues de limites à gauche.

Il établit que si les mesures $m_n = \sum_{t_i \in \tau_n} (x(t_{i+1}) - x(t_i))^2 \varepsilon_{t_i}$ où les (t_i) appartiennent à une subdivision τ_n dont le pas tend vers 0 avec n, convergent vaguement vers une mesure de fonction de répartition notée $[x,x]_t$ de partie discontinue $\sum_{s \leq t} (x(s) - x(s-))^2$, la formule d'Itô habituelle est vraie pour une fonction F de classe C^2. Il lui suffit ensuite de montrer que pour une semimartingale les trajectoires sont p.s. à variation quadratique pour une subdivision bien choisie, pour établir la formule d'Itô en toute généralité.

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References

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"Calcul d'Itô Sans Probabilités" (Föllmer 1981)

Let $X \in C^0([0,T],\mathbb{R}^d)$ and $f \in C^2(\mathbb{R}^d,\mathbb{R})$. The main idea in the proof of the Ito formula is to consider a sequence of partitions $\pi_n = (0 = t_0^n < t_1^n .. < t_{N(\pi_n)}^n = T)$ of $[0,T]$ with $|\pi_n| \to 0$ and expand increments of $f(X(t))$ along the partition using a 2nd order Taylor expansion:

$$f(X(t)) - f(X(0)) = \sum_{\pi_n} f(X(t_{i+1}^n)) - f(X(t_i^n))$$

$$= \sum_{\pi_n} \nabla f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n))$$

$$+\frac{1}{2}{}^t(X(t_{i+1}^n) - X(t_i^n))\nabla^2 f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n)) + \ r(X(t_{i+1}^n), X(t_i^n))$$

Summing over π_n we get

$$f(X(t)) - f(X(0)) = S_1(\pi_n, f) + S_2(\pi_n, f) + R(\pi_n, f)$$

- By uniform continuity of
  $r(x,y) = f(y) - f(x) - \nabla f(x).(y-x) - \frac{1}{2}{}^t(y-x)\nabla^2 f(x)(y-x)$,

  $$r(x,y) \le \varphi(||x-y||)||x-y||^2 \quad \text{with} \quad \varphi(u) \overset{u \to 0}{\to} 0$$

  $R(\pi_n, f) = \epsilon_n \sum_{\pi_n} \|X(t^n_{i+1}) - X(t^n_i)\|^2$.
- So both this term and the 'quadratic Riemann sum'

  $$S_2(\pi_n, f) = \frac{1}{2} \sum_{\pi_n} {}^t(X(t^n_{i+1}) - X(t^n_i))\nabla^2 f(X(t^n_i)).(X(t^n_{i+1}) - X(t^n_i))$$

  are controlled by the convergence of (weighted) sums of squared increments of X along π_n.

Quadratic Riemann sums

For d=1: given a path of X, pointwise convergence of 'quadratic Riemann sums'

$$S_2(\pi_n, f) = \frac{1}{2} \sum_{\pi_n} \nabla^2 f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n))^2$$

along the path for *every* $f \in C^2(\mathbb{R}^d, \mathbb{R})$ is exactly equivalent to the weak convergence of the sequence of discrete measures

$$\mu_n = \sum_{t_j \in \pi_n} (X(t_{j+1}^n) - X(t_j^n))^2 \delta_{t_j}$$

where δ_t denotes a point mass at t. This is a joint property of X and $\pi = (\pi_n)_{n \geq 1}$. This motivated Föllmer's (1981) definition of 'pathwise quadratic variation along a sequence of partitions.

Quadratic variation along a partition sequence

Definition (Föllmer 1981)

Let $\pi_n = (0 = t_0^n < t_1^n .. < t_{N(\pi_n)}^n = T)$ be a sequence of partitions of $[0, T]$ with $|\pi_n| \to 0$. A càdlàg function $x \in D([0, T], \mathbb{R})$ is said to have finite quadratic variation along the sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ if the weak limit

$$\mu := \lim_{n\to\infty} \sum_{t_j \in \pi_n} (x(t_{j+1}^n) - x(t_j^n))^2 \delta_{t_j}$$

exists and has Lebesgue decomposition given by

$$\mu([0, t]) = [x]_\pi^c(t) + \sum_{0<s\leq t} |\Delta x(s)|^2$$

where $[x]_\pi^c$ is a continuous and increasing function. $[x]_\pi(t) = \mu([0, t])$ is called the quadratic variation of x along π.

We denote $Q_\pi([0, T], \mathbb{R})$ the set of functions with this property.

Multidimensional paths

$Q_\pi([0,T],\mathbb{R}^d)$ is **not** a vector space (Schied 2016) so care must be taken in defining this notion for vector valued functions:

Definition (Paths of finite quadratic variation)

$x \in Q_\pi([0,T],\mathbb{R}^d)$ if, for all $1 \le i,j \le d$, $x^i, x^i + x^j$ in $Q_\pi([0,T],\mathbb{R})$. Then $[x]_\pi : [0,T] \to S_d^+$ defined by

$$\left([x]_\pi\right)_{i,j}(t) = \frac{1}{2}\left([x^i + x^j]_\pi(t) - [x^i]_\pi(t) - [x^j]_\pi(t)\right)$$
$$= [x^i, x^j]_\pi^c(t) + \sum_{0<s\le t} \Delta x^i(s)\Delta x^j(s), \quad i,j = 1,\dots,d$$

is an increasing function with values positive symmetric $d \times d$ matrices.

Proposition (Henry Chiu & R C (2018))

Let $x \in D([0,T],\mathbb{R}^d)$ *and define*

$$[x]_{\pi_n}(t) = \sum_{t_j \in \pi_n} (x(t^n_{j+1} \wedge t) - x(t^n_j \wedge t))^t (x(t^n_{j+1} \wedge t) - x(t^n_j \wedge t)) \in S^+_d$$

where S^+_d *is the cone of semidefinite postive symmetric* $d \times d$ *matrices The following properties are equivalent:*

1. x *has finite quadratic variation along the sequence of partitions* $(\pi_n)_{n\geq 1}$*.*
2. *The sequence* $[x]_{\pi_n}$ *converges in Skorokhod* (J_1) *topology* $[0,T]$ *to (an increasing function)* $[x]_\pi \in D([0,T],S^+_d)$*.*

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Föllmer's "pathwise Ito formula"

Proposition (Föllmer, 1981)

$\forall f \in C^2(\mathbb{R}^d, \mathbb{R}), \forall \omega \in Q_\pi([0,T], \mathbb{R}^d)$, *the non-anticipative Riemann sums along* π

$$\sum_{\pi_n} \nabla f(\omega(t_i^n)).(\omega(t_{i+1}^n) - \omega(t_i^n)) \overset{n\to\infty}{\to} \int_0^T \nabla f(\omega(t)).d^\pi\omega$$

converge pointwise and

$$f(\omega(t)) - f(\omega(0)) = \int_0^t \nabla f(\omega).d^\pi\omega + \frac{1}{2}\int_0^t < \nabla^2 f(\omega), d[\omega]_\pi^c >$$
$$+ \sum_{s\leq t} f(\omega(s)) - f(\omega(s-)) - \nabla f(\omega(s-)).\Delta\omega(s)$$

Pathwise construction of stochastic integrals

- Föllmer's result allows to define integrals of the type

$$\int \nabla f(\omega(t))d\omega \quad f \in C^2(\mathbb{R}^d, \mathbb{R})$$

  as a (pathwise) limit of (left) Riemann sums, for any path ω with finite quadratic variation along (π_n).
- Admissible integrands are gradients of C^2 functions.

Some questions:

- Extension to path-dependent integrands/ functionals? (with D Fournié, H Chiu)
- Properties of the pathwise integral: Continuity, Isometry...? (with A Ananova)
- Stability/ invariance of construction with respect to the choice of partitions? (with P Das)
- Extension to 'rougher' paths with non-zero higher-order variation? (with N Perkowski)

Invariance with respect to sequence of partitions

Joint work with Purba DAS (Oxford).

Consider two sequences of partitions π, τ and a continuous path $\omega \in Q_\pi([0,T],\mathbb{R}^d) \cap Q_\tau([0,T],\mathbb{R}^d)$. For all $f \in C^2(\mathbb{R}^d)$,

$$f(\omega(t)) - f(\omega(0)) = \int_0^t \nabla f(\omega).d^\pi\omega + \frac{1}{2}\int_0^t < \nabla^2 f(\omega), d[\omega]_\pi >$$

$$= \int_0^t \nabla f(\omega).d^\tau\omega + \frac{1}{2}\int_0^t < \nabla^2 f(\omega), d[\omega]_\tau >$$

The pathwise integrals are equal if and only if the quadratic variations along π^n and τ^n are equal : $[\omega]_\pi = [\omega]_\tau$.
Under what conditions on the path ω and the partitions is this the case?

Invariance with respect to sequence of partitions

Unfortunately pathwise quadratic variation **does** depend on the sequence of partitions...

- Freedman (1983): Let $\omega \in C^0([0,T],\mathbb{R}^d)$. There exists a sequence of partitions (π^n) such that $[\omega]_\pi = 0$.
- Davis, Obloj and Siorpaes (2018): for *any* increasing function $A : [0,T] \to [0,\infty)$ one can construct a sequence of partitions $\pi = (\pi^n)$ such that $[\omega]_\pi(t) = A(t)$.
- **On the other hand** we know that for Brownian paths, quadratic variation is a.s. equal to t along any *refining* partition (Lévy 1934) and *any* sequence of partitions with mesh $o(1/\log n)$ (Dudley 1973).
- So there must be a large class of functions which 'locally behave like Brownian motion' for which one can obtain an invariance property of quadratic variation with respect to a (large) class of partition sequences.

Well-balanced sequences of partitions

Let $\underline{\pi^n} = \inf_{i=0..N(\pi^n)-1} |t^n_{i+1} - t^n_i|$, $|\pi^n| = \sup_{i=0..N(\pi^n)-1} |t^n_{i+1} - t^n_i|$

Definition

We call the sequence of partitions $(\pi^n)_{n\geq 1}$ well-balanced if

$$\exists c > 0, \qquad \forall n \geq 1, \quad \frac{|\pi^n|}{\underline{\pi^n}} \leq c. \tag{1}$$

This condition means that the intervals in the partition π^n are asymptotically comparable.
Since $\underline{\pi^n} N(\pi^n) \leq T$, for a well-balanced sequence of partitions we have

$$|\pi^n| \leq c\underline{\pi^n} \leq \frac{cT}{N(\pi^n)}. \tag{2}$$

Averaging lemma

If X has quadratic variation along (π^n) then it also has quadratic variation along any subsequence $(\pi^{l(n)})$. Comparing sum of squares along (π^n) and the subsequence we obtain the following property:

Lemma (Cross-products of increments average to zero)

Let $X \in C^\alpha([0,T],\mathbb{R}^d)$ for some $\alpha > 0$ and $\sigma^n = \{t_i^n, i = 1..N(\sigma_n)\}$ be a well-balanced sequence of partitions of $[0,T]$ such that $X \in Q_\sigma([0,T],\mathbb{R}^d)$. Let $\kappa > \frac{1}{\alpha}$. and $(\sigma^{l_n})_{n\geq 1}$ by a subsequence of σ^n with

$$N(\sigma^{l_n - 1}) \leq N(\sigma^n)^\kappa \leq N(\sigma^{l_n}).$$

Define for $k = 1 \cdots N(\sigma^n)$, $p(k,n) = \inf\{m \geq 1 : \quad t_m^{l_n} \in (t_k^n, t_{k+1}^n]\}$. Then

$$\sum_{k=1}^{N(\sigma^n)} \sum_{p(k,n)\leq i,j \leq p(k+1,n)-1} \left(X(t_{i+1}^{l_n}) - X(t_i^{l_n})\right)^t \left(X(t_{j+1}^{l_n}) - X(t_j^{l_n})\right) \to 0.$$

Coarsening of a well-balanced partition

Coarsening of a partition corresponds to subsampling/ dropping points:

Definition (β-coarsening)

Let $\pi^n = (0 = t_0^n < t_1^n .. < t_{N(\pi^n)}^n = T)$ be a well-balanced sequence of partitions of $[0, T]$ with vanishing mesh $|\pi^n| \to 0$ and $0 \leq \beta < 1$. A β-**coarsening** of π is a sequence of subpartitions of π^n

$$A^n = (0 = t_{p(n,0)}^n < t_{p(n,1)}^n < \cdots < t_{p(n,N(A^n))}^n = T)$$

such that $(A^n)_{n \geq 1}$ is a well-balanced partition of $[0, T]$ and $|A^n| \sim |\pi^n|^\beta$.

Since $|\pi^n| \ll |A^n| \sim c|\pi^n|^\beta$, as n increases the number of points of π^n in each interval of A^n goes to infinity.

Quadratic roughness along a sequence of partitions

Let $\pi^n = (0 = t_0^n < t_1^n .. < t_{N(\pi^n)}^n = T)$ be a well-balanced sequence of partitions of $[0, T]$ with $|\pi^n| \to 0$.

Definition (Quadratic roughness along a sequence of partitions)

$X \in Q_\pi([0, T], \mathbb{R}^d)$ is $\text{quadratically rough}$ along π with coarsening index $\beta < 1$ if

- $0 < [X]_\pi < \infty$ is strictly increasing
- For any $\beta-$coarsening $(A^n)_{n\geq 1}$ of π and any $t \in (0, T]$ we have:

$$\sum_{j=1}^{N(A^n)} \sum_{t_i^n, t_{i'}^n \in [A_{j-1}^n, A_j^n)} \left(X(t_{i+1}^n \wedge t) - X(t_i^n \wedge t)\right)^t \left(X(t_{i'+1}^n \wedge t) - X(t_{i'}^n \wedge t)\right) \overset{n\to\infty}{\to} 0.$$

$R_\pi^\beta([0, T], \mathbb{R}^d) \subset Q_\pi([0, T], \mathbb{R}^d)$: set of paths with quadratic roughness property

Quadratic roughness along a sequence of partitions

Intuitively, quadratic roughness of X along $(\pi^n)_{n\geq 1}$ means that the increments of f sampled along π^n behave like a '2nd order white noise' as we refine the partition:

$$\sum_{j=1}^{N(A^n)} \sum_{t_i^n, t_{i'}^n \in [A_{j-1}^n, A_j^n)} \left(X(t_{i+1}^n \wedge t) - X(t_i^n \wedge t)\right)^t \left(X(t_{i'+1}^n \wedge t) - X(t_{i'}^n \wedge t)\right) \overset{n\to\infty}{\to} 0.$$

As expected, Brownian paths satisfy this property:

Proposition (Quadratic roughness of Brownian paths (C.-Das 2019))

Let W be a Wiener process, $T > 0$ and $0 < \beta < 1$. Then the sample paths of W almost-surely satisfy the quadratic roughness property with coarsening index β along any well-balanced partition sequence with mesh $o(1/(\log n)^{1/\beta})$:

$$|\pi^n| = o(1/(\log n)^{1/\beta}) \Rightarrow \mathbb{P}\left(W \in R_\pi^\beta([0,T])\right) = 1.$$

Invariance of quadratic variation for rough functions

Our main result is that quadratic roughness implies invariance of pathwise quadratic variation along well-balanced sequences of partitions:

Theorem (Cont & Das (2019))

Let $X \in C^{\alpha}([0,T],\mathbb{R}^d) \cap Q_{\sigma}([0,T],\mathbb{R}^d)$ for some well-balanced sequence of partitions $\sigma = (\sigma^n)$ of $[0,T]$ with $\limsup |\sigma^n|/|\sigma^{n+1}|$¡$\infty$.If $x \in R^{\alpha}_{\sigma}([0,T])$ then for any other well-balanced sequence of partitions $\tau = (\tau^n)_{n\geq 1}$,

- *If $x \in Q_{\tau}([0,T],\mathbb{R}^d)$ then $[x]_{\sigma} = [x]_{\tau}$.*
- *There exists a subsequence $\pi^n = \tau^{k(n)}$ of τ such that $x \in Q_{\pi}([0,T],\mathbb{R}^d)$ and $[x]_{\sigma} = [x]_{\pi}$.*

Extension to non-anticipative Functionals

Denote $\omega_t = \omega(t \wedge .)$ the *past* i.e. the path stopped at t.

Definition (Non-anticipative Functionals)

A *causal*, or *non-anticipative functional* is a functional $F : [0, T] \times D([0, T], \mathbb{R}^d) \mapsto \mathbb{R}$ whose value only depends on the past:

$$\forall \omega \in \Omega, \quad \forall t \in [0, T], \qquad F(t, \omega) = F(t, \omega_t). \tag{3}$$

Causal functional= map on the space Λ_T^d of stopped paths, defined as the quotient space:

$$\Lambda_T^d := \left([0, T] \times D([0, T], \mathbb{R}^d)\right) \Big/ \sim$$

where $(t, x) \sim (t', x') \leftrightarrow t = t', x_t = x'_t$. Λ_T^d is equipped with a metric

$$d_\infty((t, x), (t', x')) = \sup_{u \in [0, T]} |x(u \wedge t) - x'(u \wedge t')| + |t - t'|.$$

$\mathbb{C}^{0,0}(\Lambda_T^d)$ = continuous maps $(\Lambda_T^d, d_\infty) \to \mathbb{R}$.

Definition (Horizontal and vertical derivatives)

A non-anticipative functional F is said to be:

- **horizontally differentiable** at $(t, \omega) \in \Lambda_T^d$ if the finite limit exists

$$\mathcal{D}F(t, \omega) := \lim_{h \to 0+} \frac{F(t+h, \omega_t) - F(t, \omega_t)}{h}.$$

- vertically differentiable at $(t, \omega) \in \Lambda_T^d$ if the map

$$\mathbb{R}^d \to \mathbb{R}, \; e \mapsto F(t, \omega(t \wedge .) + e 1_{[t,T]})$$

  is differentiable at 0; its gradient at 0 is denoted by $\nabla_\omega F(t, \omega)$.

Note that $\mathcal{D}F(t, \omega)$ is **not** the partial derivative in t:

$$\mathcal{D}F(t, \omega) \neq \partial_t F(t, \omega) = \lim_{h \to 0} \frac{F(t+h, \omega) - F(t, \omega)}{h}.$$

Smooth functionals

Definition ($\mathbb{C}^{1,p}_b(\Lambda^d_T)$ functionals)

We denote by $\mathbb{C}^{1,p}_b(\Lambda^d_T)$ the set of non-anticipative functionals $F \in \mathbb{C}^{0,0}_l(\Lambda^d_T)$, such that

- F is horizontally differentiable with $\mathcal{D}F$ continuous at fixed times,
- F is p times vertically differentiable with $\nabla^j_\omega F \in \mathbb{C}^{0,0}_l(\Lambda^d_T)$ for $j = 1..p$
- $\mathcal{D}F, \nabla^j_\omega F \in \mathbb{B}(\Lambda^d_T)$ for $j = 1..p$.

Example (Cylindrical functionals)

For $g \in C^0(\mathbb{R}^{d\times n}), h \in C^k(\mathbb{R}^d)$ with $h(0) = 0$. Then

$$F(t,\omega) = h\left(\omega(t) - \omega(t_n-)\right) \quad 1_{t\geq t_n} \quad g(\omega(t_1-), \omega(t_2-)..., \omega(t_n-)) \qquad \text{is } \mathbb{C}^{1,k}_b$$

$$\mathcal{D}_t F(\omega) = 0, \quad \nabla^j_\omega F(t,\omega) = h^{(j)}(\omega(t) - \omega(t_n-))1_{t\geq t_n} g(\omega(t_1-), ..., \omega(t_n-))$$

$\mathbb{S}(\Lambda_T, \pi_n) :=$ simple predictable cylindrical functionals along π_n,
$\mathbb{S}(\Lambda_T, \pi) := \cup_{n>1} \mathbb{S}(\Lambda_T, \pi_n)$x

Examples of smooth functionals

Example (Integral functionals)

For $g \in C_0(\mathbb{R}^d)$, $Y(t) = \int_0^t g(X(u))\rho(u)du = F(t, X_t)$ where

$$F(t,\omega) = \int_0^t g(\omega(u))\rho(u)du \tag{4}$$

$F \in \mathbb{C}_b^{1,\infty}$, with:

$$\mathcal{D}_t F(\omega) = g(\omega(t))\rho(t) \qquad \nabla_\omega^j F(t,\omega) = 0 \tag{5}$$

Functional change of variable formula: $p = 2$

Theorem (R.C.- Fournié ,2010)

Let $\omega \in V_2(\pi) \cap C^0([0,T],\mathbb{R}^d)$ *and* $\omega^n := \sum_{i=0}^{m(n)-1} \omega(t_{i+1}^n -)\mathbf{1}_{[t_i^n, t_{i+1}^n)} + \omega(T)\mathbf{1}_{\{T\}}$. *Then for any* $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$, *the pointwise limit of Riemann sums*

$$\int_0^T \nabla_\omega F(t,\omega) d^\pi \omega \quad := \lim_{n\to\infty} \sum_{i=0}^{m(n)-1} \nabla_\omega F(t_i^n, \omega^n)(\omega(t_{i+1}^n) - \omega(t_i^n))$$

exists and

$$F(T,\omega) = F(0,\omega) + \int_0^T \nabla_\omega F(t,\omega)\cdot d^\pi\omega + \int_0^T \mathcal{D}F(t,\omega)dt + \int_0^T \frac{1}{2} tr\left(\nabla_\omega^2 F(t,\omega) d[\omega]_\pi^c(t)\right).$$

These result allows to construct $\int_0^{\cdot} \nabla_\omega F$ as a pointwise limit of non-anticipative 'Riemann sums':

$$\int_0^T \nabla_\omega F(t, \omega_{t-}) \cdot d^\pi \omega = \lim_{n\to\infty} \sum_{i=0}^{m(n)-1} \nabla_\omega F(t_i^n, \omega^n)(\omega(t_{i+1}^n) - \omega(t_i^n))$$

Remark

$$F \in \mathcal{R}(\Lambda_T^d),\ \omega \in C^{\frac{1}{2}-}([0,T], \mathbb{R}^d) \Rightarrow \nabla_\omega F(t, \omega) \in C^{\frac{1}{2}-}([0,T], \mathbb{R}^d).$$

The pathwise integral is a **strict extension** *of the Young integral.*

Denote $C^{\frac{1}{2}-}([0,T],\mathbb{R}^d) = \bigcap_{\nu<1/2} C^{\nu}([0,T],\mathbb{R}^d)$

Theorem (Pathwise Isometry formula, A. Ananova, R. C. 2016)

Let $\omega \in Q_\pi([0,T],\mathbb{R}^d) \cap C^{1/2-}([0,T],\mathbb{R}^d)$ where $|\pi^n| \to 0$. If $F \in \mathbb{C}^{1,2}(\Lambda_T)$ is Lipschitz-continuous and $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T^d)$ then

$$F(.,\omega) \in Q_\pi([0,T],\mathbb{R}^d), \qquad \int_0^{\cdot} \nabla_\omega F(.,\omega).d^\pi\omega \in Q_\pi([0,T],\mathbb{R}^d)$$

$$[F(t,\omega)]^\pi(t) = \left[\int_0^{\cdot} \nabla_\omega F(s,\omega).d^\pi\omega\right]^\pi(t) = \int_0^t \langle {}^t\nabla_\omega F(s,\omega).\nabla_\omega F(s,\omega), d[\omega]^\pi(s)\rangle.$$

This is a **pathwise version of the Ito isometry formula**: if we take expectations on both sides with respect to the Wiener measure we recover the well-known Ito isometry.

Conditional expectations as smooth functionals

Vertical smoothness: $H : (D([0,T],R), \|.\|_\infty) \mapsto \mathbb{R}$ is vertically smooth on continuous paths if for $(t,\omega) \in [0,T] \times C^0([0,T],\mathbb{R}^d)$, the map

$$g^H(.;t,\omega) : e \in \mathbb{R}^d \to \quad g^H(e) = H\left(\omega + e1_{[t,T]}\right)\right), \tag{6}$$

is twice differentiable at 0, with derivatives bounded in a neighborhood of 0, uniformly with respect to $(t,\omega) \in [0,T] \times C^0([0,T],\mathbb{R}^d)$.

Proposition (C & Riga 2016)

Let $\mathbb{Q}$ be the Wiener process and $H : (D([0,T],R), \|.\|_\infty) \mapsto \mathbb{R}$ be $\mathbb{Q}$-integrable, Lipschitz and vertically smooth. Then there exists $F \in \mathbb{C}_b^{0,2}(\mathcal{W}_T)$ such that $F(t,X_t) = E^{\mathbb{Q}^\sigma}[H(X)|\mathcal{F}_t^X]$ $\mathbb{Q}^\sigma - a.s.$

El Karoui, Jeanblanc and Shreve (1997) revisited

Proposition (Pathwise analysis of hedging strategies)

Let $\omega \in Q_\pi([0, T], \mathbb{R}_+)$ be a strictly positive price path whose quadratic variation $[\omega]$ along $\pi = (\pi_n)$ is absolutely continuous.

$$\text{Denote} \qquad \sigma(t,\omega)^2 = \frac{1}{\omega(t)^2}\frac{d[\omega]}{dt}$$

If there exists $F \in \mathbb{C}^{1,2}(\mathcal{W}_T)$ such that

$$F(t,\omega) = E[H|S_t = \omega_t]$$

A delta-hedging strategy for H computed in a Black-Scholes model with volatility σ_0 leads to a profit/loss along the path ω given by

$$\int_0^T \frac{\sigma_0^2 - \sigma^2(t,\omega)}{2}\omega(t)^2 e^{\int_t^T r(s)ds}\underbrace{\nabla_\omega^2 F(t,\omega)}_{\Gamma(t)}dt$$

p-th variation along a sequence of partitions

Define the *oscillation* of $S \in C([0,T],\mathbb{R})$ along π_n as

$$osc(S,\pi_n) := \max_{[t_j,t_{j+1}]\in\pi_n} \max_{r,s\in[t_j,t_{j+1}]} |S(s)-S(r)|.$$

Definition (p-th variation along a sequence of partitions)

Let $p>0$. Let $S \in C([0,T],\mathbb{R})$ with $osc(S,\pi_n)\to 0$. The sequence of measures

$$\mu^n = \sum_{[t_j,t_{j+1}]\in\pi_n} \delta(\cdot - t_j)|S(t_{j+1})-S(t_j)|^p$$

converges weakly to a measure μ without atoms $\iff$ There exists a continuous increasing function $[S]^p$ such that

$$\forall t\in[0,T], \qquad \sum_{\pi_n} |S(t_{j+1}\wedge t)-S(t_j\wedge t)|^p \overset{n\to\infty}{\to} [S]^p(t).$$

Then $[S]^p(t) := \mu([0,t])$ and we write $S\in V_p(\pi)$ and $[S]^p(t) := \mu([0,t])$ is the *p-th variation* of S along π.

p-th variation along a sequence of partitions

Lemma

Let $S \in C([0,T],\mathbb{R})$. $S \in V_p(\pi)$ if and only if there exists a continuous increasing function $[S]^p$ such that

$$\forall t \in [0,T], \qquad \sum_{\pi_n} |S(t_{j+1} \wedge t) - S(t_j \wedge t)|^p \overset{n\to\infty}{\to} [S]^p(t).$$

If this property holds then the convergence is uniform.

Examples of processes with sample paths in $V_p(\pi)$

Paths in $V_p(\pi)$ do not necessarily have finite p-**variation**.

- If B_H is a **fractional Brownian motion** B_H with Hurst index $H \in (0,1)$ and $\pi_n = \{kT/n : k \in \mathbb{N}_0\} \cap [0,T]$, then $B_H \in V_{1/H}(\pi)$ and $[B_H]^{1/H}(t) = t\mathbb{E}[|B_H(1)|^{1/H}]$, while $\|B_H\|_{p-var} = \infty$ almost surely for $p = 1/H$.
- **Stochastic heat equation**: if $u(t,x)$ is the solution of the stochastic heat equation with white noise on $[0,T] \times \mathbb{R}$ then $u(.,x) \in V_4([0,T])$, while $\|u(.,x)\|_{4-var} = \infty$.
- **Reflected Brownian motion in wedge** (limit process arising in queueing theory).

'Rough' Change of variable formula

Theorem (R.C- Perkowski (2018))

Let $p \in 2\mathbb{N}$ be even, $S \in V_p(\pi)$. Then for every $f \in C^p(\mathbb{R}, \mathbb{R})$

$$f(S(t)) - f(S(0)) = \int_0^t < \nabla_{p-1} f(S), dS > + \frac{1}{p!} \int_0^t f^{(p)}(S(s)) d[S]^p(s),$$

where the integral is defined as a (pointwise) limit of compensated Riemann sums:

$$\int_0^t \nabla_{p-1} f \circ S.dS := \int_0^t < \nabla_{p-1} f(S)(u), dS(u) >$$

$$= \lim_{n \to \infty} \sum_{\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!} (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k$$

In particular we construct a pathwise Ito-type integral + change of variable formula for FBM with *any* Hurst exponent.

Pathwise integral

The pathwise integral

$$\int_0^t < \nabla_{p-1} f \circ S, dS >:= \lim_n R_{p-1}(f, S, \pi_n)$$

is a pointwise limit of compensated Riemann sums

$$R_{p-1}(f, S, \pi_n) = \sum_{\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!} (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k$$

It should be really seen as an integral of the $(p-1)-$**jet** $\nabla_{p-1} f$ of f

$$\nabla_{p-1} f(x) = (f^{(k)}(x), k = 0, 1, ..., p-1)$$

with respect to a differential structure of order $p-1$ constructed along $S \in V_p(\pi)$ using the powers of increments up to order $p-1$.

The following result extends the pathwise isometry formula obtained in (Ananova-C. 2017) for $p = 2$:

Theorem (Isometry formula (C.-Perkowski 2018))

Let $p \in 2\mathbb{N}$ be an even integer, (π_n) a sequence of partitions with mesh size going to zero, and $S \in V_p(\pi) \cap C^\alpha([0,T],\mathbb{R})$ for some $\alpha > 0$. Then for any $f \in C^p(\mathbb{R}^d)$,

$$f \circ S \in V_p(\pi) \qquad \int_0^{\cdot} \nabla_{p-1} f \; dS := \int_0^{\cdot} < \nabla_{p-1} f(S), dS > \in V_p(\pi)$$

$$[f(S)]^p(T) = [\int_0^{\cdot} \nabla_{p-1} f \; dS]^p(T) = \int_0^T |f'(S)|^p d[S]^p = \|f' \circ S\|^p_{L^p([0,T],d[S]^p)}.$$

Pathwise local time of order p

Definition (Local time of order p)

Let $p \in \mathbb{N}$ be an even integer and let $q \in [1, \infty]$. A continuous path $S \in C([0,T], \mathbb{R})$ has an L^q*-local time of order* $p-1$ along a sequence of partitions $\pi = (\pi_n)_{n \geq 1}$ if $osc(S, \pi_n) \to 0$ and

$$L_t^{\pi_n, p-1}(\cdot) = \sum_{t_j \in \pi} \mathbf{1}_{(\![S(t_j \wedge t), S_{t_{j+1} \wedge t}]\!]}(\cdot) |S(t_{j+1} \wedge t) - \cdot|^{p-1}$$

converges weakly in $L^q(\mathbb{R})$ to a weakly continuous map $L : [0,T] \to L^q(\mathbb{R})$ which we call the *order p local time* of S. We denote $\mathcal{L}_p^q(\pi)$ the set of continuous paths S with this property.

Intuitively, the limit $L_t(x)$ then measures the rate at which the path S accumulates p-th order variation near x.

Theorem (Higher order Tanaka-Wuermli formula)

Let $p \in 2\mathbb{N}$ be an even integer, $q \in [1, \infty]$ with conjugate exponent $q' = q/(q-1)$. Let $f \in C^{p-1}(\mathbb{R}, \mathbb{R})$ and assume that $f^{(p-1)}$ is weakly differentiable with derivative in $L^{q'}(\mathbb{R})$. Then for any $S \in \mathcal{L}_p^q(\pi)$

$$\int_0^t \nabla_{p-1} f \circ S \mathrm{d}S := \lim_{n\to\infty} \sum_{[t_j, t_{j+1}] \in \pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!} (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k$$

exists and the following change of variable formula holds:

$$f(S(t)) - f(S(0)) = \int_0^t < \nabla_{p-1} f \circ S, dS > + \frac{1}{(p-1)!} \int_{\mathbb{R}} f^{(p)}(x) L_t(x) \mathrm{d}x.$$

Multidimensional paths: Symmetric tensors

A symmetric p-tensor T on $\mathbb{R}^d$ is a p-tensor invariant under any permutation $\sigma \in \mathfrak{S}_p$ of its arguments: for $(v_1, v_2, \ldots, v_p) \in (\mathbb{R}^d)^p$

$$\sigma T(v_1, \ldots, v_p) := T(v_{\sigma 1}, v_{\sigma 2}, \ldots, v_{\sigma p}) = T(v_1, v_2, \ldots, v_p)$$

The space $\mathrm{Sym}_p(\mathbb{R}^d)$ of symmetric tensors of order p on $\mathbb{R}^d$ is naturally isomorphic to the dual of the space $\mathbb{H}_p[X_1, ..., X_d]$ of homogeneous polynomials of degree p on $\mathbb{R}^d$.

$$\mathbb{S}_p(\mathbb{R}^d) = \bigoplus_{k=0}^{p} \mathrm{Sym}_k(\mathbb{R}^d).$$

For any p-tensor T we define the *symmetric part*

$$\mathrm{Sym}(T) := \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_k} \sigma T \in \mathrm{Sym}_p(\mathbb{R}^d)$$

where $\mathfrak{S}_p$ of $\{1, \ldots, k\}$ is the group of permutations of $\{1, 2, ..., p\}$

Extension to multidimensional functions

Consider now a continuous $\mathbb{R}^d$-valued path $S \in C([0,T],\mathbb{R}^d)$ and a sequence of partitions $\pi_n = \{t_0^n, \ldots, t_{N(\pi_n)}^n\}$ with $t_0^n = 0 < ... < t_k^n < ... < t_{N(\pi_n)}^n = T$. Then

$$\mu^n = \sum_{\pi_n} \underbrace{(S(t_{j+1}) - S(t_j)) \otimes ... \otimes (S(t_{j+1}) - S(t_j))}_{\text{p times}} \delta(\cdot - t_j)$$

defines a tensor-valued measure on $[0,T]$ with values in $\mathrm{Sym}_p(\mathbb{R}^d)$. This space of measures is in duality with the space $C([0,T],\mathbb{H}_p[X_1,...,X_d])$ of continuous functions taking values in homogeneous polynomials of degree p i.e. homogeneous polynomials of degree p with continuous time-dependent coefficients.
This motivates the following definition:

Definition (p-th variation of a multidimensional function)

Let $p \in 2\mathbb{N}$ be an (even) integer, and $S \in C([0,T],\mathbb{R}^d)$ a continuous path and $\pi = (\pi_n)_{n\geq 1}$ a sequence of partitions of $[0,T]$. $S \in C([0,T],\mathbb{R}^d)$ is said to have a p-th variation along $\pi = (\pi_n)_{n\geq 1}$ if $osc(S,\pi_n) \to 0$ and the sequence of tensor-valued measures

$$\mu_S^n = \sum_{\pi_n} (S(t_{j+1}) - S(t_j))^{\otimes p} \quad \delta(\cdot - t_j)$$

converges to a $\mathrm{Sym}_p(\mathbb{R}^d)$-valued measure μ_S without atoms in the following sense: $\forall f \in C([0,T],\mathbb{S}_{\mathbf{p}}(\mathbb{R}^{\mathbf{d}}))$,

$$< f, \mu_n > = \quad \sum\nolimits_{\pi_n} < f(t_j), (S(t_{j+1}) - S(t_j))^{\otimes p} > \to^{n\to\infty} < f, \mu_S > . \qquad (7)$$

We write $S \in V_p(\pi)$ and call $[S]^p(t) := \mu([0,t])$ the *p-th variation* of S.

Theorem (Rough change of variable formula: multi-dim case)

Let $p \in 2\mathbb{N}$ be an even integer, let (π_n) be a sequence of partitions of $[0, T]$ and $S \in V_p(\pi) \cap C([0, T], \mathbb{R}^d)$. Then for every $f \in C^p(\mathbb{R}, \mathbb{R})$ the limit of compensated Riemann sums

$$\int_0^t < \nabla_{p-1} f \circ S, dS > := \lim_{n \to \infty} \sum_{\pi_n} \sum_{k=1}^{p-1} \frac{1}{k!} < \nabla^k f(S(t_j)), (S(t_{j+1} \wedge t) - S(t_j \wedge t))^{\otimes k} >$$

exists for every $t \in [0, T]$ and satisfies

$$f(S(t)) - f(S(0)) = \int_0^t < \nabla_{p-1} f \circ S, dS > + \frac{1}{p!} \int_0^t < \nabla^p f(S(t))), d[S]^p(u) > .$$

Functional change of variable formula: general case

Theorem (C.- Perkowski, 2018)

Let $p \in 2\mathbb{N}$ $F \in \mathbb{C}_b^{1,p}(\Lambda_T)$*, and* $S \in V_p(\pi)$ *for a sequence of partitions* (π_n) *with* $|\pi_n| \to 0$*. Then the limit* $\int_0^t < \mathbb{T}_{p-1}F(.,S), dS >=$

$$\lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{1}{k!}\nabla_\omega^k F(t_j, S^n_{t_j-})(S(t_{j+1}\wedge t) - S(t_j\wedge t))^k, \quad \text{exists and}$$

$$F(t,S_t) = F(0,S_0) + \int_0^t \mathcal{D}F(s,S_s)ds$$

$+ \int_0^t < \mathbb{T}_{p-1}F(.,S),\ dS > + \frac{1}{p!}\int_0^t \nabla_\omega^p F(s,S_s)d[S]^p(s)$

This extends the pathwise integral to all 'closed $(p-1)$ forms':
$\mathbb{T}_{p-1}\mathbb{C}_b^{1,p} := \{\mathbb{T}_{p-1}F,\ F \in \mathbb{C}_b^{1,p}(\Lambda_T)\}$

A higher order isometry formula

The following result extends the isometry formula for the pathwise integral obtained in the case $p = 2$ by (Ananova-C. 2017):

Theorem (Isometry formula)

Let $p \in \mathbb{N}$ be an even integer, (π_n) a sequence of partitions with mesh size going to zero, and $S \in V_p(\pi) \cap C^\alpha([0,T],\mathbb{R})$ with $\alpha > ((1+\frac{4}{p})^{1/2} - 1)/2$. Let $F \in \mathbb{C}_b^{1,p}(\Lambda_T) \cap \mathrm{Lip}(\Lambda_T, d_\infty)$ be such that $\nabla_\omega F \in \mathbb{C}_b^{1,p-1}(\Lambda_T)$. Then

$$F(\cdot, S) \in V_p(\pi), \qquad \int_0^{\cdot} \mathbb{T}_{p-1} F(\cdot, S) dS \in V_p(\pi) \qquad \text{and}$$

$$[\int_0^{\cdot} \mathbb{T}_{p-1} F(\cdot, S) dS]^p(t) = \int_0^t |\nabla_\omega F(s, S)|^p d[S]^p(s) = \|\nabla_\omega F(., S)\|^p_{L^p([0,T],d[S]^p)}.$$

Rough-smooth decomposition

'Signal+noise' decomposition for smooth functionals of a rough process:

Theorem (Rough-smooth decomposition: general case)

Let $p \in \mathbb{N}$ be an even integer, let $\alpha > ((1+\frac{4}{p})^{1/2}-1)/2$, and let $S \in V_p(\pi) \cap C^\alpha([0,T],\mathbb{R})$ be a path with strictly increasing p-th variation $[S]^p$ along (π_n). Then any $X \in \mathbb{C}_b^{1,p}(S)$ admits a unique decomposition

$$\exists ! \phi \in \mathbb{T}_{p-1}\mathbb{C}_b^{1,p}, \qquad X = X(0) + A + \int_0^t < \phi \circ S, dS >$$

where ϕ is a closed $(p-1)$-form and $[A]^p = 0$.

- For S martingale, $p=2$ this is a 'Doob Meyer' decomposition. However our formulation is strictly pathwise/ non-probabilistic.
- Such pathwise decompositions were obtained in the rough path setting by Hairer & Pillai (2013), Friz & Shekhar (2013).

Relation with 'rough path integration'

Define a *control function* as a continuous map $c\colon \Delta_T \to \mathbb{R}_+$ such that $c(t,t) = 0$ and $c(s,u) + c(u,t) \leq c(s,t)$.

Definition (Reduced rough path of order p)

Let $p \geq 1$. A *reduced rough path* of finite p-variation is a map
$\mathbb{X} = (1, \mathbb{X}^1, \ldots, \mathbb{X}^{\lfloor p \rfloor})\colon \Delta_T \longrightarrow \mathbb{S}_{\lfloor p \rfloor}(\mathbb{R}^d)$,
such that

$$\sum_{k=1}^{\lfloor p \rfloor} |\mathbb{X}^k_{s,t}|^{p/k} \leq c(s,t), \qquad (s,t) \in \Delta_T;$$

for some control function c and the *reduced Chen relation* holds

$$\mathbb{X}_{s,t} = \mathrm{Sym}(\mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t}), \qquad (s,u),(u,t) \in \Delta_T.$$

A canonical reduced rough path for $S \in V_p(\pi)$

Lemma

Let $p \geq 1$, $S \in C([0,T], \mathbb{R}^d) \cap V_p(\pi)$ *where*

$$\pi_n = (t^n_k), \qquad t^n_0 = 0, \qquad t^n_{k+1} = \inf\{t \in [t^n_k, T], \quad |S(t) - S(t^n_k)| \geq 2^{-n}\}.$$

Then for any $q > p$ *with* $\lfloor q \rfloor = \lfloor p \rfloor$ *we obtain a reduced rough path of finite* q*-variation by setting* $\mathbb{X}^0_{s,t}(S) := 1$,

$$\mathbb{X}^k_{s,t}(S) := \frac{1}{k!}(S(t) - S(s))^{\otimes k}, \qquad k = 1, \ldots, \lfloor p \rfloor - 1,$$
$$\mathbb{X}^{\lfloor p \rfloor}_{s,t}(S) := \frac{1}{\lfloor p \rfloor!}(S(t) - S(s))^{\otimes \lfloor p \rfloor} - \frac{1}{\lfloor p \rfloor!}([S]^p(t) - [S]^p(s)).$$

Furthermore $\mathbb{X} : S \mapsto \mathbb{X}$ *is a non-anticipative functional.*

Proposition

Let $p \geq 1$, let $\mathbb{X}$ be a reduced rough path of finite p-variation and let $Y \in \mathcal{D}_{\mathbb{X}}^{\lfloor p \rfloor / p}([0, T])$. Then the 'rough path integral'

$$I_{\mathbb{X}}(Y)(t) = \int_0^t \langle Y(s), \mathrm{d}\mathbb{X}(s) \rangle = \lim_{\substack{\pi \in \Pi([0,t]) \\ |\pi| \to 0}} \sum_{[t_j, t_{j+1}] \in \pi} \sum_{k=1}^{\lfloor p \rfloor} \langle Y^k(t_j), \mathbb{X}^k_{t_j, t_{j+1}} \rangle,$$

defines a function in $C([0, T], \mathbb{R})$, and it is the unique function with $I_{\mathbb{X}}(Y)(0) = 0$ for which there exists a control function c with

$$\Big| \int_s^t \langle Y(r), \mathrm{d}\mathbb{X}(r) \rangle - \sum_{k=1}^{\lfloor p \rfloor} \langle Y^k(s), \mathbb{X}^k_{s,t} \rangle \Big| \lesssim c(s, t)^{\frac{\lfloor p \rfloor + 1}{p}}, \qquad (s, t) \in \Delta_T.$$

Pathwise integral as canonical rough integral

Proposition (C- Perkowski 2018)

Let $p \in 2\mathbb{N}$ be an even integer, $S \in V_p(\pi)$ and $\mathbb{X}$ the canonical reduced rough path of order p associated to S, defined above. Then

$$\underbrace{\int_0^t \langle \nabla f(S(s)), \mathrm{d}\mathbb{X}(s)\rangle}_{\text{Rough integral}} = \underbrace{\int_0^t \langle \nabla_{p-1} f(S), \mathrm{d}S\rangle}_{\text{Pathwise integral}},$$

where the right hand side is the pathwise integral defined as a limit of compensated Riemann sums.