# Recover Dynamic Utility from Observed Characteristic Process
# An Application to the Economic Equilibrium

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# Decision making in economics and finance

*Gomez, Inverse problem in Economics (2012),*

Decision making under uncertainty

- Most of decision making focuses on the derivation of the optimal process and its out-comes given preferencies.
- In finance the preference are based on expected utility criterium.
- Available observed data are the result of the decision process, and its dynamics over the time.

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*Dybvig&Rogers: Recovery of Preference from single realization of wealth,1995*

Theory of revealed preference: an inverse problem ?

- question: what "observed data " tell us about preferences?
- The results address both the existence and uniqueness of a preference in a minimally restricted class.
- Poor literature is related to the dynamics of both optimal and preferences problems.

# Example: Basic static optimization problem

## Static problem with budget constraint (BC)

- A given *a priori utility* $v$ of the final "wealth"
- A convex (cone) family $\mathscr{X}$ of random (wealth ) variables $X \geq 0$
- At least a pricing kernel $Y$ with $\mathbb{E}(Y) \leq 1$ ($= 1$ "complete market)
- $\bar{u}(x) = \max\{\mathbb{E}(v(X))|X \in \mathscr{X}\}$, when (BC) $\mathbb{E}(Y\,X) \leq x$

## The equivalent dual problem based on optimal choice of $Y$

- Let $\mathscr{Y}$ be orthogonal convex cone of $\mathscr{X}$; $\mathbb{E}(YX) \leq x, \mathbb{E}(Y) \leq 1$
- The dual problem is $\tilde{u}(y) = \min_Y\{\mathbb{E}(\tilde{v}(yY)|\mathbb{E}(YX) \leq 1\}$ with $(x = 1)$.
- $\tilde{u}(y)$ is the convex conjuguate of $\bar{u}(x)$

## Links between the optimal solutions

If $\exists$ a state price s.t. $\mathbb{E}(Y^*) = 1$, and $-\tilde{v}'_y(yY^*) \in \mathscr{X}$,

- then the optimum is $X^*(x) = -\tilde{v}_y(y(x)Y^*)$ , where
- $y(x)$ is selected to achieve the budget constraint, if it is possible

$$\mathbb{E}[-\tilde{v}_y(y(x)Y^*)Y^*] = x$$

## A priori and A posteriori utilities

Value function $\bar{u}(x) = \max\{\mathbb{E}(v(X))|X \in \mathscr{X} \text{ s.t. } \mathbb{E}(YX) \leq x\}$

- Put $\bar{u}(x) = \max\{\mathbb{E}(v(x\bar{X}))|\bar{X} \in \mathscr{X}, \text{s.t. } \mathbb{E}(Y\bar{X}) \leq 1\}$,
- $\bar{u}$ is a regular utility function, the *a posteriori utility value function*
- At the optimum, $v_z(X^*(x)) = y(x)Y^*$,
  - $X^*(x)$ is increasing, and $\mathbb{E}(v(X^*(x))) = \bar{u}(x)$
  - $x\,\bar{u}_x(x) = \mathbb{E}(X^*(x)v_z(X^*(x))) = y(x)\mathbb{E}(X^*(x)Y^*) = x\,y(x)$

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The inverse problem:

$$\boxed{\text{To recover } v \text{ from } (\bar{u},\ X^*(x) \text{ and } Y^*) \text{ for increasing } X^*(x)}$$

- Let $\mathcal{X}(z) = (X^*)^{-1}(z)$ be the inverse r.v. of $x \mapsto X^*(x)$.
  If $\bar{u}_x(\mathcal{X}(z))$ is integrable near to 0,
- Thanks to the necessary condition $v(X^*(x)) = \bar{u}(x)Y^*$, the answer is obvious, (for eventually random utility $v$)

$$v_z(z) = \bar{u}_x(\mathcal{X}(z))Y^*, \quad v(x) = \int_0^x \bar{u}_x(\mathcal{X}(z))Y^* dz, \quad (v(0) = 0)$$

# Dynamic Utilities and their characteristics

Dynamic Utility Framework on $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$

- A dynamic utility $U(t,z)$ is a family of càdlàg adapted regular utility functions, with Fenchel conjugate $\widetilde{U}(t,y)$.
- A characteristic process $(X_t^c(x))$ is an adapted stochastic flow
  - x-increasing, "optimal" in the sense
  - $U(t, X_t^c(x))$ is a martingale
- In addition, if $X_t^c(x)$ is an "optimal" choice in a family $\mathscr{X}$, then $U(t, X_t)$ is a supermartingale $\forall X \in \mathscr{X}$.

Conjuguate Utility $(\widetilde{U}(t,y))$ when no optimization problem

- Put $Y_t^c(u_x(x)) = U_z(t, X_t^c(x))$ the dual characteristic process
- By the master equation
  $U(t,z) - zU_z(t,z) = \widetilde{U}(t, U_z(t,z)) \quad \text{and} \quad \widetilde{U}_y(t, U_z(t,z)) = -z$
- $\widetilde{U}(t, Y_t^c(u_x(x)))$ is a "martingale" if and only if $X_t^c(x) Y_t^c(u_x(x))$ is a "martingale"

# Definition of the recovery problem

The processes, observable $\mathbf{X}^c$, adjoint $\mathbf{Y}$, and the initial utility $u(0,z)$

- the observable process $(X_t^c(x))$ is an increasing $x$-continuous optional flow $(X_t^c(x))$ with $(X_t^c(0)) = 0$, with range $(0,\infty)$ and optional inverse $\mathcal{X}_t^c(z) = (X_t^c)^{-1}(z)$
- $(Y_t(y))$ is an intermediate increasing $y$-continuous optional process with range $(0,\infty)$, called adjoint process
- $u$ is a regular utility

Forward Utility construction

Necessarily the stochastic utility $U(t,z)$ verifies

- $U_z(t,z) = Y_t(u_z(\mathcal{X}_t^c(z)))$, $\quad U(t,x) = \int_0^x Y_t(u_z(\mathcal{X}_t^c(z)))\,dz$
- $U(t, X_t^c(x)) = \int_0^x Y_t(u_x(z))\,d_z X_t^c(z)$ is a "martingale".
- These last integral is a Stieljes integral, with explosion near to $z = 0$.

**General framework, no reference to any financial market, no regularity in time.**

# Definition of the recovery problem

**Definition** (Compatible utility).

Let $(\mathbf{X}^c, \mathcal{X}^c) \in \mathfrak{I} \times \mathfrak{I}$ be an increasing observable process and its inverse, and $u$ the initial utility.
A dynamic utility $\mathbf{U}$ is said to be compatible with $(X^c, u)$ if and only if there exists an admissible adjoint process, $\mathbf{Y} \in \mathfrak{I}(X^c, u)$ satisfying the *"first order condition"*:

$$U_z(t,z) = Y_t(u_x(\mathcal{X}^c_t(z)) \text{ or } U(t, X^c_t(x)) = \int_0^x Y_t(u_x(z)) d_z X^c_t(z).$$

The class of compatible dynamic utilities is denoted $\mathfrak{U}(X^c, u)$.

- Given $(\mathbf{X}^c, u)$, there is a one to one correspondance between the classes of compatible utilities $\mathfrak{U}(X^c, u)$ and admissible adjoint processes $\mathfrak{I}(X^c, u)$.

**Definition** (Revealed utility).

A compatible dynamic utility $U$ is $\mathbf{U} \in \mathfrak{U}(X^c, u)$ is said to be a $(X^c, u)$-revealed dynamic utility if and only if:

$$\forall x \in (0, \infty), \quad U(t, X_t^c(x)) \text{ is a positive martingale}$$

Or equivalently, its associated adjoint process $\mathbf{Y} \in \mathfrak{I}(X^c, u)$ satisfies the condition "$\int_0^x Y_t(u_x(z)) d_z X_t^c(z)$ is a martingale". The class of revealed dynamic utilities is denoted $\mathfrak{U}_{\mathfrak{M}}(X^c, u)$ and the class of such adjoint processes is denoted $\mathfrak{I}_{\mathfrak{M}}(X^c, u)$.

Req: $\forall \mathbf{U} \in \mathfrak{U}_{\mathfrak{M}}(X^c, u)$, $(\tilde{U}(t, Y_t(y)))$ is a martingale if and only if:
$\forall x \in (0, \infty)$, $X_t^c(x)\, Y_t(u_z(x))$ is a positive martingale.

# Example: Linear (in $x$) characteristic process

## Constant characteristic portfolio

- Prop: A forward utility $U(t,z)$ is a martingale if and only if the marginal utility $U_z(t,x)$ is a martingale
  - thanks to the inequality $z\,U_z(t,z) \leq U(t,z) \leq U(t,z_{\mathbf{max}})$.
  - By the Lebesgue derivative theorem, $U_z(t,z) = Y_t(u_z(z))$ is a martingale dominated by the martingale $U(t,z)/z$.
  - Conversely, if $U_z(t,z)$ is a martingale, that is also true for any $z_{\mathbf{0}} > 0$ for $U(t,z) - U(t,z_{\mathbf{0}})$, and by monotony in $z_{\mathbf{0}}$ for $U(t,z)$.

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## Linear characteristic process $X_t^c(x) := xX_t^c(1) = x\,X_t$

- Use $X_t$ as numeraire and define $U^X(t,z) = U(t,zX_t)$, which is a martingale with characteristic process $x$
- By the previous result, the condition is equivalent to "$U_z^X(t,z) = X_t\,U_z(t,zX_t) = X_t\,Y_t(u_z(z))$ is a martingale".
- Then, $U_z(t,z) = Y_t(u_z(z/X_t))$, and if $u$ is a power utility, then $U$ is a power utility, if and only if $Y$ is linear.

# Differentiable characteristic

**Proposition.**

Let $\mathbf{U} \in \mathfrak{U}(X^c, u)$ be a dynamic utility with adjoint process $\mathbf{Y}$, whose the characteristic process $\mathbf{X}^c$ is $x$-differentiable with derivative $\{X^c_x(t,x)\}$.

(i) If the characteristic process is convex ($x \to X^c_x(t,x)$ positive increasing), then $\mathbf{U}$ is a revealed utility if and only if $\{X^c_x(t,x)Y_t(u_z(x)\}$ is a martingale for any $x$.

(ii) In the general case, the condition is only sufficient; if $\{X^c_x(t,x)Y_t(u_z(x)\}$ is a martingale then $\{U(X^c_t(x))\}$ is a martingale.

(i) A particular case of the previous example, $\{U^X(t,z) := U(t, X^c_t(z))\}$ is a martingale dynamic utility, which is equivalent, by previous example, to the martingale property of $\{U^X_z(t,z) = Y_t(u_z(z))X^c_x(t,z)\}$.

(ii) Let $\Psi^X(t,x_0,x) := \int_{x_0}^x Y_t(u_z(z))X^c_x(t,z)dz$, which is a martingale with expectation $u(x) - u(x_0)$, by the positive (Fubini). By monotony and positivity, this property goes to the limit when $x_0 \to 0 \Rightarrow U(t,x) := \Psi^X(t, \mathcal{X}^c(t,x))$ is a revealed dynamic utility.

# Existence of revealed utility, general case

Darboux Approximation of $\int_0^x Y_t^c(u_x(z))\, d_z X_t^c(z) = U(t, X_t^c(x))$

- Let $0 < x_1 \ldots x_n < x_{n+1} \ldots < x$, and $\xi_n(t)$ a r.v. $x_n \leq \xi_n(t) \leq x_{n+1}$,
- $S_N^{\xi}(t,x) = \sum_{n=0}^{n=N-1} Y_t^c(u_z(\xi_n(t)))(X_t^c(x_{n+1}) - X_t^c(x_n))$
- By Young's Theorem (1936), $S_N^{\xi}(t,x)$ converges a.s. to $U(t,x)$

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**Theorem 1** (Necessary and Sufficient Condition).

An utility $U$ is a revealed utility if and only if $\exists$ a sequence $(x_n \leq \xi_n(t,x) \leq x_{n+1})$ s.t. $S_N^{\xi}(t,x)$ is a martingale.

# Supermartingale conditions

In the main result, the existence of a process $\psi_t(z, z')$ can be difficult to establish.

**Theorem 2.**

Let $\mathbf{U} \in \mathfrak{U}(X^c, u)$ be dynamic utility with adjoint process $Y$. Assume that for $x, x' > 0$, the positive process $\{Y_t(u_z(x))(X_t^c(x') - X_t^c(x))\}$ is a supermartingale, then the dynamic utility $\mathbf{U}$ is a revealed utility.

# Consistent Dynamic Utility

**Definition**

Let $\mathscr{X}$ be a convex cone of non negative processes called Test processes

- An $\mathscr{X}$- consistent dynamic utility $U(t,x)$ is a progressive utility s.t
  - For any test process $X \in \mathscr{X}$, $U(t, X_t)$ is a supermartingale
  - For any initial wealth $x > 0$, there exists a characteristic test process $X^c \in \mathscr{X}, X^c_0 = x$, such that $U(t, X^c_t(x))$ is a martingale

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**Main Question**

- under which conditions on $(\mathscr{X}, u, X^c, Y)$ a revealed utility $U$ is a $\mathscr{X}$-consistent utility
- Another formulation, under which conditions, the revealed utility $U$ is the value function of an optimization problem defined on $\mathscr{X}$, with random final utility $U(T_H, z)$
- Links to financial market

# A General Market Model I

Incomplete Market :

- Let $W$ be a $n$-Brownian motion, a short rate process $r_t$ and a risk premium vector $\eta_t$, and
- $\mathscr{X}$ is the class of (positif) wealth processes $X^\kappa$ driven by the self-financing equation

$$dX_t^\kappa = X_t^\kappa \left[ r_t dt + \kappa_t.(dW_t + \eta_t^\sigma dt) \right], \ \ \eta_t^\sigma, \kappa_t \in \mathcal{R}_t^\sigma$$

  - $\sigma_t$ is the $dxn$ volatility matrix, and $\sigma_t.\sigma_t^\top$ is invertible.
  - Let $\pi_t$ be the wealth proportions invested in the different assets, and $\kappa_t = \sigma_t \pi_t$,
  - Constraints: $\mathcal{R}_t^\sigma$ is a family of adapted subvector spaces in $\mathbb{R}^n$, typically $\mathcal{R}_t^\sigma = \sigma_t(\mathbb{R}^d)$, $d \leq n$.
  - $\eta_t^\sigma \in \mathcal{R}_t^\sigma$ defined as the projection of $\eta_t$ on $\mathcal{R}_t^\sigma$ is the minimal risk premium,

All processes are adapted with good integrability properties

# A General Market Model, II

Adjoint processes and state price processes

- A process $Y$ is to be a strong adjoint process or a a state price process is if for any $\kappa \in \mathcal{R}^{\sigma}$, $Y_{.} X^{\kappa}_{.}$ is a martingale.
- $Y$ and $X^{\kappa}$ are strongly orthogonal, i.e., $Y(y)X^{\kappa}(x)$ is a martingale $\forall x, y$.

Characterization

- there exist $\nu \in \mathcal{R}^{\sigma,\perp}$ : $\frac{dY^{\nu}_t}{Y^{\nu}_t} = -r_t dt + (\nu_t - \eta^{\sigma}_t).dW_t, \ \nu_t \in \mathcal{R}^{\sigma,\perp}_t$
- $\mathcal{R}^{\sigma,\perp}$ is the orthogonal cone of $\mathcal{R}^{\sigma}$

Old Existence Result Under regularity assumptions on the stochastic flows $X^c$ and $Y^c$, we have shown the existence and uniqueness of revealed consistent utility, with identification of its Itô's decomposition, in terms of stochastic PDE: Heavy computational

## Economic equilibrium: H. He & H. Leland [HL93] framework

Equilibrium strategy in Markovian complete market from He and Leland

- Stock diffusion: $dS_t = S_t\,(\mu(t,S_t)dt + \sigma(t,S_t).dW_t)$
- complete market with given interest rate $r_t$
- the risk premium is $\sigma(t,S_t)\eta(t,S_t) = \mu(t,S_t) - r_t$
- In complete market, the adjoint process is $dY^e_t = -Y^e_t\big(\mu(t,S_t)dt + \eta_t(S_t)(dW_t - \sigma(t,S_t).dt)\big)$
- The equilibrium strategy is $X^c_t(x) = S_t(x)$, monotonic in $x$ if coefficients are regular in $x$.

The forward point of view with given initial utility $u$

- The adjoint process is the process $Y^e$ with initial condition $u_z$
- The forward utility applied to $X^c$ is $U(t, X^c_t(x)) = Y^e_t(u_z(x))$
- In the Markovain case, (He and Leland), the previous relation implied SPDE type constraints on the coefficients of SDE $Y^e$, similar to our paper on Itô case.

**Proposition.**

The dual equilibrium revealed problem admits a solution iff:

- The pricing kernel is a geometrical Brownian motion, that is $\partial_y \zeta(t, y) = 0$.

$$Y^e_t(y) := yY^e_t = y \exp\Big(-\int_0^t r_s ds - \int_0^t \zeta_s dW_s - \frac{1}{2}\int_0^t \zeta_s^2 ds\Big)$$

- The "pricing" PDE

$$\partial_t \Phi(t, y) + \frac{1}{2} y^2 \zeta_t^2 \Phi_{yy}(t, y) - y r_t \Phi_y(t, y) = 0 \qquad (1)$$

admits a positive convex decreasing solution, such that the local martingale $\{\Phi(t, Y^e_t(y))\}$ is a "true" martingale.

Observe that the convexity and decreasing in space implies the decreasing in time of the solutions of (1).

**Proposition.**

Assume that $\forall t \int_0^t \zeta_s^2 ds < \infty$, and $\zeta_t^2 > 0$. Let $\mu(d\beta)$ be a positive Borel measure defined on $(1, \infty)$, such that $\int_1^\infty \frac{y^{\mathbf{1}-\beta}}{\beta-\mathbf{1}} \mu(d\beta) = \phi^\mu(y) < \infty$.

The function $y \to \phi^\mu(y)$ is the initial condition (in time) of the family of time-dependent dual utility, $\{\Phi^{(\mu)}(t, y)\}$ positive solution of the PDE (1), where

$\Phi^{(\mu)}(t, y) = \int_1^\infty H(t, \beta) \frac{y^{\mathbf{1}-\beta}}{\beta-\mathbf{1}} \mu(d\beta)$ and $\{\Phi^{(\mu)}(t, Y_t^e(y))\}$ is a positive martingale. Moreover, by Widder's results [1963], such functions are the only possible convex solutions.

# Pareto Optimality and Sup-convolution

Since all agents have the same optimal dual process, the equilibrium is Pareto optimal.

**Theorem 3.**

Let $U$ be the dynamic utility of a representative agent. A economic equilibrium holds if and only if there exists a positive Borel measure $\mu$ such that,

- The utility process $U$ is given as the sup-convolution:

$$U(t,x) = \sup\{\int_1^\infty U^{(\beta)}(t, x^\beta(x))\mu(d\beta);\ \int_1^\infty x^\beta(x)\mu(d\beta) = x\}$$

- The supremum is achieved at $\{x^\beta(t,x) := (U_z^{(\beta)})^{-1}(t, U_z(t,x)),\ \beta\}$.
- $U$ is a revealed stochastic utility with optimal portfolio $S_t^{(\mu)}(x) = \int_1^\infty X^{*,\beta}(t, x^\beta(x))\mu(d\beta)$. Moreover, for any $\beta$, the optimal wealth is $x^\beta(t, S_t^{(\mu)}(x)) = X^{*,\beta}(t, x^\beta(x))$.
- $U$ is the value function of a forward problem, that is $U(t, X_t(x))$ is a supermartingale for any $X$ solution of a dynamics of the form $dX_t(x) = X_t(x)\big(r_t dt + \kappa(t, X_t(x))(dW_t + \zeta_t dt\big)$.

Thank You for your attention!

# Merci Nicole