Recent progresses of LLN and CTL under Uncertainty

Peng Shige

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I.I.D. assumption in statistics and econometrics

- I.I.D. assumptions for sequences of real world data $\{X_i\}_{i=1}^{\infty}$;
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- Worst case philosophy:

$$\hat{\mathbb{E}}[X] := \max_{\theta\in\Theta} E_{P_\theta}[X], \qquad -\hat{\mathbb{E}}[-X] := \min_{\theta\in\Theta} E_{P_\theta}[X],$$

$\hat{\mathbb{E}}$ is a sublinear expectation. Inversely...

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From linear to Nonlinear Expectations

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  - $\mathbb{E}[X_i] \downarrow 0$, if $X_i(\omega) \downarrow 0$, $\forall \omega$

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$$\mathbb{F}[\varphi] = \sup_{\theta \in \Theta} F_\theta[\varphi] = \sup_{\theta \in \Theta} \int_\Omega \varphi(X) dP_\theta$$

Uncertainty version of I.I.D. of two random variables X and Y

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$$(\mathit{resp}.\mathbb{E}[\varphi(X)] \geq \mathbb{E}[\varphi(Y)], \quad \text{denoted by} \quad X \overset{d}{\geq} Y\ )$$

- Y is indenpendent of X if for any test function $\varphi(x, y)$,

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}].$$

Uncertainty version of I.I.D. time sequence $\{X_i\}_{i=1}^{\infty}$

$$\mathbb{E}[\varphi(X_i)] = \mathbb{E}[\varphi(X_1)], \quad X_{i+1} \text{ is independent of } (X_1, \cdots, X_i), \quad \text{for all } i.$$

Remark.

$$\{X_i\}_{i=1}^{\infty} \text{ is an IID} \quad \Longrightarrow \quad \{\varphi(X_i)\}_{i=1}^{\infty} \quad \text{is also IID} \quad \forall \varphi$$

□

A typical toy model: nonlinear Bernoulli model

- At time $t = 1$, we randomly choose a ball from an urn containing Black and White balls ($W_1 + B_1 = 100$)
- But we know only $W_1 \in [40, 60]$
- $$X_1(\omega) = \mathbf{1}_{\{W_1 = \text{true}\}} - \mathbf{1}_{\{B_1 = \text{true}\}}$$

Repeat this game at $t = 1, 2, 3, \cdots,$

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Repeat this game at $t = 1, 2, 3, \cdots,$

- Each time i, W_i is changed within $W_i \in [40, 60]$
- We get a sequence of random variables $\{X_i(\omega)\}_{i=1}^{\infty}$;
- This is a typical case of our daily uncertainty: enveronment changes all the time
- The output $\{X_i\}_{i=1}^{\infty}$ is IID:
  - $X_1, X_2, \cdots$ are identically distributed (same distribution uncertainty)
  - X_{i+1} is independent of $\{X_i\}_{i=1}^{n}$.
- $S_n = \sum_{i=1}^{n} X_i$: a nonlinear Bernoulli random walk.
- $\{\varphi(X_i)\}_{i=1}^{\infty}$ is also IID, for any given function $\varphi(x)$.

A general random generator of nonlinear IID sequence $\{X_i\}_{i=1}^{\infty}$

- The urn $\Longrightarrow$ A generator of random vectors X_i, at time $t = i$,

$$X_i(\omega) \overset{d}{=} F_\theta, \quad i = 1, 2, 3, \cdots$$

We can observe the output X_i at $t = i$ with

$$\mathcal{L}(X_i) \in \{F_\theta\}_{\theta \in \Theta}$$

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An important special case: I.I.D. maximal distributed sequence $\{X_i\}_{i=1}^{\infty}$

- $\{F_\theta\}_{\theta\in\Theta} = M_{[\underline{\mu},\overline{\mu}]}$: all prob. distributions concentrated on $[\underline{\mu},\overline{\mu}]$.
- Many cases, we have

$$\mathcal{L}(X_i) \in \{F_\theta\}_{\theta\in\Theta_i}, \quad \Theta_i \subset \Theta$$

But we use $X_i \overset{d}{=} \{F_\theta\}_{\theta\in\Theta}$ to robustly cover the uncertainty.

Important advantage of nonlinear expectation space

- **Example**: Maximally-distributed i.i.d sequence $\{X_i\}_{i=1}^{\infty}$: covers all possible distributions of the sequence satisfying

$$\underline{\mu} \leq X_i(\omega) \leq \overline{\mu}, \quad i = 1, 2, \cdots \quad \mathbb{E}[\varphi(X_i)] = \max_{v \in [\underline{\mu}, \overline{\mu}]} \varphi(v)$$

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- How to narrow down $\overline{\mu} - \underline{\mu}$ through real data is our important task
- Challenging objective: to establish a systematic framework $(\Omega, \mathcal{H}, \mathbb{E})$ compatible with $(\Omega, \mathcal{F}, P)$
- Important: hidden behind are PDEs (linear/nonlinear heat equation)!

Two fundamentally important nonlinear distributions

- $Y \overset{d}{=} M_{[\underline{\mu},\overline{\mu}]}$ is defined by $aY + b\bar{Y} \overset{d}{=} (a+b)Y$;
- $X \overset{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ is defined by $aX + b\bar{X} \overset{d}{=} \sqrt{a^2+b^2}X$;

The universality of the IID assumption under sublinear expectation

- For a bounded random sequence $\{X_i\}_{i=1}^{\infty}$ one can always enlarge the probability uncertainty so that $\{X_i\}_{i=1}^{\infty}$ is an I.I.D. sequence under the enforced sublinear expectation $\mathbb{E}$.

Lemma

Assume that a random sequence $\{X_i\}_{i=1}^{\infty}$ is bounded by $\underline{\mu} \leq X_i \leq \overline{\mu}$. Then we have $X_i \overset{d}{\leq} \bar{X}_i$ and $\{X_i\}_{i=1}^{\infty} \overset{d}{\leq} \{\bar{X}_i\}_{i=1}^{\infty}$, where $\{\bar{X}_i\}_{i=1}^{\infty}$ is an IID sequence with $\bar{X}_i \overset{d}{=} M_{[\underline{\mu},\overline{\mu}]}$.

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Thus one can robustly enforce $\mathbb{E}[\cdot]$ and assume that $\{X_i\}_{i=1}^{\infty}$ is IID under $\mathbb{E}[\cdot]$.

The universality of the IID assumption under sublinear expectation

- If a random sequence $\{X_i\}_{i=1}^{\infty}$ is not bounded, we can set

$$Y_i^{(j)} = \frac{2}{\pi} \arctan(X_i^{(j)}), \quad j = 1, 2, \cdots, d.$$

  Then $\{Y_i\}_{i=1}^{\infty}$ can be robustly assumed to be IID.
- On the other hand, the 'IID' assumption is very flexible and can cover many important cases.
- **Example** If $\{X_i\}_{i=1}^{\infty}$ is i.i.d. and the distribution of X_1 is linear, then it becomes to IID sequence in the classical case.

Nonlinear Law of large number

Theorem (Peng2007)

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$$\lim_{n\to\infty}\mathbb{E}[\varphi(\frac{Y_1+\cdots+Y_n}{n})]=\mathbb{E}[(\varphi(Y)]=\max_{v\in[\underline{\mu},\overline{\mu}]}\varphi(v).$$

where $\overline{\mu}=\mathbb{E}[Y_1]$, $\underline{\mu}=-\mathbb{E}[-Y_1]$.

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$u(x,t):=\mathbb{E}[\varphi(x+(1-t)Y)]$ solves the 1st order PDE

Nonlinear Central limit theorem

Theorem (Peng2008-2010)

Let $\{X_i\}_{i=1}^{\infty}$ *be IID sequence. Assume furthermore*

$$\mathbb{E}[(|X_1|^{2+\epsilon}] < \infty \quad \mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0$$

Then, for each $\varphi \in C_b(\mathbb{R})$,

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\frac{X_1 + \cdots + X_n}{\sqrt{n}})] = u^{\varphi}(0,0) = \mathcal{N}_G(\varphi).$$

$$\partial_t u^{\varphi} + G(\partial^2_{xx} u^{\varphi}) = 0, \quad t \in [0,1], \quad u^{\varphi}(1,x) = \varphi(x),$$

$$G(a) := \frac{1}{2}[\overline{\sigma}^2 a^+ - \overline{\sigma}^2 a^-]$$

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Sketch of the proof: $u \in C^{1+\frac{\alpha}{2},2+\alpha}$ [Krylov]

$$\lim_{n\to\infty} \mathbb{E}[\varphi(S_n^n)] = u^{\varphi}(0,0),$$

$S_k^n := \frac{1}{\sqrt{n}}(X_1 + \cdots + X_k)$, $k = 1, \cdots, n$.

$$u(S_n^n, 1) - u(0,0) = \sum_{k=0}^{n-1} [u(S_{k+1}^n, \frac{k+1}{n}) - u(S_k^n, \frac{k}{n})]$$

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$$\mathbb{E}[u(S_{k+1}^n, \frac{k+1}{n}) - u(S_k^n, \frac{k}{n})] = \mathbb{E}[u(S_k^n + \frac{1}{\sqrt{n}}X_{k+1}, \frac{k}{n} + \frac{1}{n}) - u(S_k^n, \frac{k}{n})]$$

$$= \mathbb{E}[\frac{1}{n}\partial_t u(x,t) + \frac{X_{k+1}}{\sqrt{n}}\partial_x u(x,t) + \frac{X_{k+1}^2}{2n}\partial_{xx}^2 u(x,t)] + o(\frac{1}{n})$$

$$= 0 + o(\frac{1}{n}), \qquad (x,t) = (S_k^n, \frac{k}{n})$$

Convergence rate of LLN

Theorem (Fang-Peng-Shao-Song(2017))

We have

$$\mathbb{E}[d^2_{[\underline{\mu},\overline{\mu}]}(\overline{X}_n)] = \mathbb{E}[((\overline{X}_n - \overline{\mu})^+)^2 + ((\overline{X}_n - \underline{\mu})^-)^2] \leq \frac{2[\overline{\sigma}^2 + (\overline{\mu} - \underline{\mu})^2]}{n}, \quad (1)$$

where

$$\overline{\mu} = \mathbb{E}[X_1], \ \underline{\mu} = -\mathbb{E}[-X_1].$$

and

$$\overline{\sigma}^2 := \sup_{\theta \in \Theta} E_{P_\theta}[(X_1 - E_{P_\theta}[X_1])^2].$$

Converges rate of nonlinear LLN and CLT

Remark. ([Fang-P.-Shao,Song 2017])

If $\overline{\mu} > \underline{\mu}$ then the convergence of $\frac{1}{n}(X_1 + \cdots + X_n))$ to cannot be a strong one!

[Fang-P.-Shao,Song] (2017) Limit theorems with rate of convergence under sublinear expectations.

Stein Method under nonlinear expectation

Stein equation: the key tool of Stein method. **Song (2017) had found the corresponding "Stein equation", and provided Nonlinear Stein method.**

the rate of convergence is::

$$\sup_{|\varphi|_{Lip}\leq 1}\left|\hat{\mathbf{E}}[\varphi(\frac{X_1+\cdots+X_n}{\sqrt{n}})]-\mathcal{N}_G(\varphi)\right|\lesssim\frac{1}{n^{\frac{\alpha}{2}}},$$

where $\alpha\in(0,1)$ depends only on $-\hat{\mathbf{E}}[-X_1^2]$ and $\hat{\mathbf{E}}[X_1^2]$.

Converges rate of nonlinear CLT

Theorem (Krylov 2018)

Assume that $|\phi(x) - \phi(y)| \leq |x-y|^\beta, \quad M_\beta := \sup_{\xi \in \Theta} E(|\xi|^{2+\beta}) < \infty.$
Then

$$|\mathbb{E}[\varphi(\tfrac{1}{n}(Y_1 + \cdots + Y_n))] - \mathbb{E}[\varphi(Y)]| \leq Nn^{-\beta^2/(4+2\beta)}$$

where N *depends only on* M_β *and* $\underline{\sigma}^2$.

- [Jin &Peng] (2016) Optimal unbiased estimation for maximal distribution, arXiv:1611.07994v1,
- [Fang-Peng.-Shao & Song] (2017) Limit theorems with rate of convergence under sublinear expectations, arxiv.
- [Song] (2017) Normal approximation by Stein's method under sublinear expectations",
- [Krylov 2018] "On Shige Peng's Central Limit Theorem" , arxiv

Related works

- [Maccheroni & Marinacci] (2005) "A strong law of large numbers for capacities", Ann. Probab. 33, 1171-1178.
- [Marinacci] (1999) "Limit laws for non-additive probabilities and their frequentist interpretation", J. Econom. Theory.

Nonlinear normal distributions

Definition

A random variable X in $(\Omega, \mathcal{H}, \mathbb{E})$ is normal if the function

$$u(t,x) := \mathbb{E}[\varphi(x+\sqrt{1-t}X)]$$

is the solution of the nonlinear PDE

$$\partial_t u + G(\partial_{xx} u) = 0, \;\; u(1,x) = \varphi(x).$$

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- $G(a) = \frac{1}{2}[\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-]$ $\quad \overline{\sigma}^2 = \mathbb{E}[X^2], \ \ \underline{\sigma}^2 = -\mathbb{E}[-X^2]$

Classical 'Monté-Carlo' approach for estimating $\hat{\mathbb{E}}[\varphi(X)]$ through data

- Key point: How to obtain $\hat{\mathbb{E}}[\varphi(X)]$ through its sample $\{x_i\}_{i=1}^N$?
- In many practice cases: we care about $\hat{\mathbb{E}}[\varphi(X)]$ with a specific function $\varphi(x)$:
  a consumption utility function, a contract, a cost function ....
- In a classical probability space $(\Omega, \mathcal{F}, P)$, we can apply LLN to calculate

$$E[\varphi(X)] \sim \mathbb{M}[\varphi(X)] := \frac{1}{N}\sum_{i=1}^{N} \varphi(x_i)$$

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- But: Is $\{x_i\}_{i=1}^N$ a classical IID?

φ-max-mean algorithm: the data-based distribution of X

- Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space and

$$\{x_i\}_{i=1}^{n\times m}: \quad \text{IID sample of a random vector} \quad X$$

- The max-mean algorithm to estimate $\hat{\mathbb{E}}[\varphi(X)]$:

$$\hat{\mathbb{M}}[\varphi] = \max\{Y_n^k : k = 0, \cdots, m-1\},$$

where

$$Y_n^k = \frac{1}{n}\sum_{i=1}^{n} \varphi(x_{nk+i}).$$

- By the above LLN, as $n \to \infty$, $\{Y_n^k\}_{k=0}^{m-1} \overset{d}{\Longrightarrow}$ an IID $\{Y^k\}_{k=0}^{m-1}$,

$$Y^k \overset{d}{=} M_{[\underline{\mu},\overline{\mu}]}, \quad \text{with} \quad \overline{\mu} = \hat{\mathbb{E}}[\varphi(X)], \ \underline{\mu} = -\hat{\mathbb{E}}[-\varphi(X)]$$

- But $\max\{Y_n^k : k = 0, \cdots, m-1\}$ provides us the asymptotically optimal unbiased estimate .

φ-max-mean algorithm of X from IID sample $\{x_i\}_{i=1}^{mn}$

$$\max\left\{\underbrace{\frac{\varphi(X_1)+\cdots+\varphi(X_n)}{n}}_{Y_n^0},\cdots,\underbrace{\frac{\cdots}{n}}_{Y_n^k},\cdots,\underbrace{\frac{\varphi(X_{(m-1)n+1})+\cdots+\varphi(X_{mn})}{n}}_{Y_n^{m-1}}\right\}$$

$$\begin{aligned}\mathbb{E}[\varphi(X)] &\simeq \text{Max-Mean-}[\varphi(\{x_i\})] \\ &= \max_{0\le k\le m-1} \frac{\sum_{i=1}^{n}\varphi(x_{kn+i})}{n}\end{aligned}$$

Optimality of the estimate

The optimality of the above estimate is based on the following quite simple, but very fundamental result:

Theorem (Jin-Peng2016)

Let $Y^1, \cdots, Y^m$ be IID and maximally distributed:

$$Y^i \overset{d}{=} M_{[\underline{\mu},\overline{\mu}]}, \qquad i = 1, \cdots, m,$$

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$$\underline{\mu} \leq \min\{Y^1(\omega), \cdots, Y^n(\omega)\} \leq \max\{Y^1(\omega), \cdots, Y^n(\omega)\} \leq \overline{\mu}.$$

Moreover

$$\widehat{\overline{\mu}}_n = \max\{Y^1, \cdots, Y^n\},$$

is the maximum unbiased estimate of $\overline{\mu}$.

Parameter estimates for nonlinear distributed model

- Many typical nonlinear distributions

$$M_{[\underline{\mu},\overline{\mu}]}, \quad N(\mu, [\underline{\sigma}^2, \overline{\sigma}^2]), \quad P_{[\underline{\lambda},\overline{\lambda}]} \text{ (Nonlinear Poisson)}$$

- asymptotically unbiased estimates: ,

$$\hat{\underline{\sigma}}^2 := \min_{1\leq k\leq m} \sigma_k^2, \qquad \hat{\overline{\sigma}}^2 := \max_{1\leq k\leq m} \sigma_k^2$$

$$\text{where} \quad \sigma_k^2 := \frac{1}{n}\sum_{j=1}^{n}(x_{n(k-1)+j} - \mu)^2.$$

Maximal and Normal distributions and Nonlinear PDEs

- $Y \stackrel{d}{=} M_{[\underline{\mu},\overline{\mu}]}$ defined by $aY + b\bar{Y} \stackrel{d}{=} (a+b)Y$;
  is directly related to the 1st order PDE

$$\partial_t u^\varphi + g(\partial_x u^\varphi) = 0,$$
$$u^\varphi(x,1) = \varphi(x).$$

  for $u^\varphi(t,x)$ on $t \in [0,1]$, $x \in \mathbb{R}^d$.

- Through

$$\mathbb{F}_g[\varphi] := u^\varphi(0,0) = \mathbb{E}[\varphi(Y)].$$

Maximal and Normal distributions and Nonlinear PDEs

- $X \stackrel{d}{=} N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ defined by $aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2+b^2}X$ is calculated by the 2nd order parabolic PDEs

$$\partial_t v^\varphi + G(\partial^2_{xx} v^\varphi) = 0,$$
$$v^\varphi(x, 1) = \varphi(x).$$

Through :

$$\mathbb{F}_G[\varphi] := v^\varphi(0,0) = \mathbb{E}[\varphi(X)]$$

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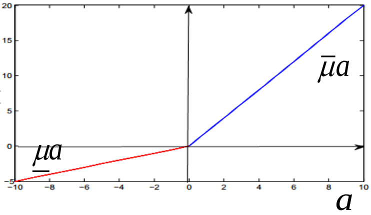

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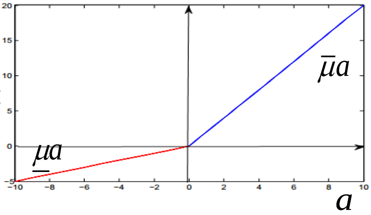

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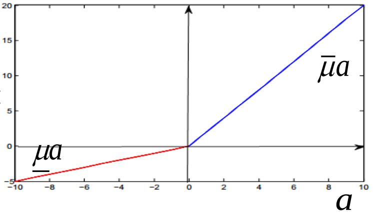

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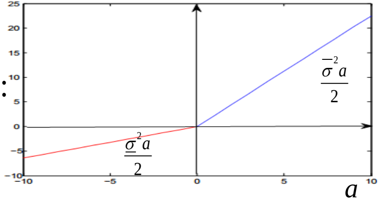

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- P. (2010) Nonlinear Expectations and Stochastic Calculus under Uncertainty (2010a). Preprint: arXiv:1002.4546 [math.PR]
- Zhang (2015) Donsker's invariance principle under the sublinear expectation with an application to Chung's law of the iterated logarithm. Commun. Math. Stat .

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Theorem.

If $(B_t)_{t\geq 0}$ is a G–**Brownian motion** and $\mathbb{E}[B_t] = \mathbb{E}[-B_t] \equiv 0$ then:
$B_{t+s} - B_s \overset{d}{=} N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, $\forall\ s, t \geq 0$ □

Real case study: from VaR to GVaR Problem challenged by CFFEX (China Financial Future Exchange)

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F_G has the explicit expression:

$$F_G(x) = \int_{-\infty}^{x} \frac{\sqrt{2}}{\sqrt{\pi(\overline{\sigma}+\underline{\sigma})^2}} \left[\exp\left(\frac{-y^2}{2\overline{\sigma}^2}\right)\mathbf{1}_{y\le 0} + \exp\left(\frac{-y^2}{2\underline{\sigma}^2}\right)\mathbf{1}_{y>0}\right] dy. \tag{4}$$

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- Fix a window width $w = 100$ use the moving window

$$\sigma_{\bar{t},w}^2 := \sigma^2(X_{\bar{t}-w+1}, \cdots, X_{\bar{t}}).$$

- Then get the upper and low data variances:

$$\overline{\sigma}_{\bar{t}}^2 = \max\{\sigma_{\bar{t},20}^2, \sigma_{\bar{t}-s,w}^2,;\ s \in [0,\cdots,l-w]\},$$
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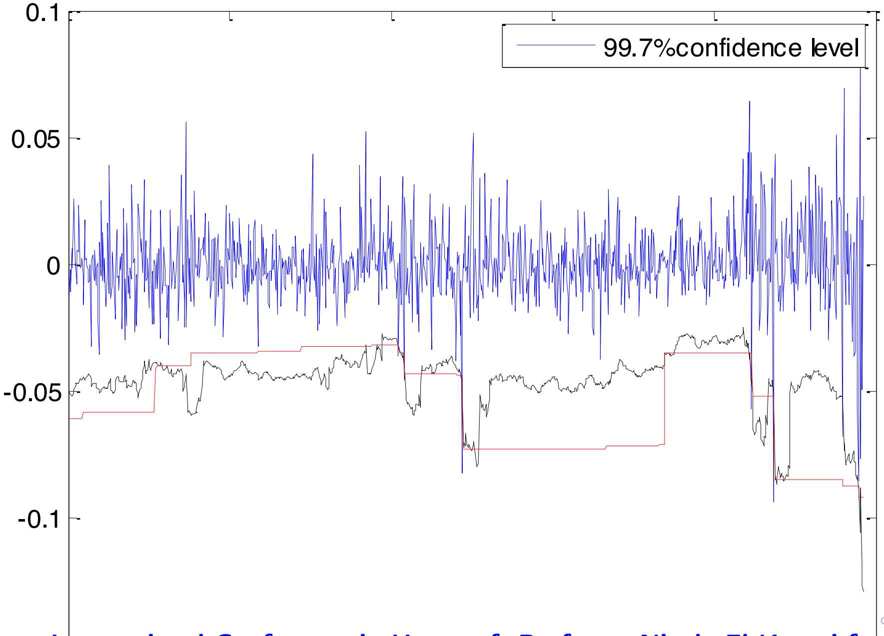
99.7%confidence level
0.1
0.05
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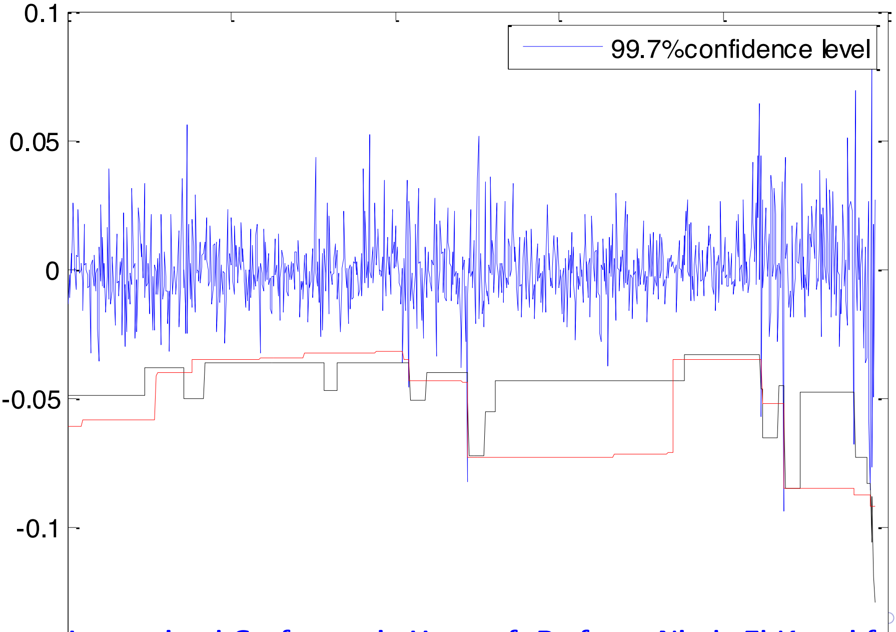

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Comparison:

K. Kuester, S. Mittnik, and M. S. Paolella. Value-at-Risk Prediction: A compar- ison of alternative strategies. Journal of Financial Econometrics, 4(1):53–89, 2006.

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- Maximal distribution and nonlinear normal distribution is fundamental, and can quantitatively cover most real world cases.
- The max-mean algorithm, based on the nonlinear LLN gives us the asymptotical optimal estimate of the nonlinear mean $\mathbb{E}[\varphi(X)]$ of X through its real data sample $\{x_i\}$ is very robust.
- This algorithm provide us automatically the degree of uncertainty, through the degree of its nonlinearity.

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- According to our new law of large number, maximal distribution $M_{[\underline{\mu},\overline{\mu}]}$ is the other typical case which was often treated as a constant.
- Need a deep collaboration of experts from probability, functional analysis, PDE and stochastic PDE, scientific computing, especially with experts of economics and statistics to develop this new, deep directions
- Combining with machine learning, to get more robust and deeper understanding the information we can obtain through a real and dynamical sample $\{x_i\}$.

A Nicole

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