Recent progresses of LLN and CTL under Uncertainty

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I.I.D. assumptions for sequences of real world data \( \{X_i\}_{i=1}^{\infty} \);

- Machine learning, deep learning, risk measuring pricing in finance ...
- Black-Scholes pricing formula, model uncertainty
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- Use nonlinear expectation, specially sublinear expectation,
- Covering probability uncertainty \( \{P_\theta\}_{\theta \in \Theta} \)
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- Covering probability uncertainty \( \{P_\theta\}_{\theta \in \Theta} \)
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- Worst case philosophy:

\[
\hat{E}[X] := \max_{\theta \in \Theta} E_{P_\theta}[X], \quad -\hat{E}[-X] := \min_{\theta \in \Theta} E_{P_\theta}[X],
\]

\( \hat{E} \) is a sublinear expectation. Inversely...
From Probability to probability uncertainty
From linear to Nonlinear Expectations

- $(\Omega, \mathcal{F}, P)$ Probability space $\iff$ Linear Expectation

Nonlinear expectation $E: H \mapsto \mathbb{R}$
- Monotonicity: $E[X] \geq E[Y]$ if $X \geq Y$
- Constant preserving: $E[c] = c$
- Sublinearity: $E[X + Y] \leq E[X] + E[Y]$ and $E[\lambda X] = \lambda E[X]$, $\forall \lambda \geq 0$
- $E[X_i] \downarrow 0$, if $X_i(\omega) \downarrow 0$, $\forall \omega$
From Probability to probability uncertainty
From linear to Nonlinear Expectations

- $(\Omega, \mathcal{F}, P)$ Probability space $\iff$ Linear Expectation
- $(\Omega, \mathcal{H}, \mathbb{E})$, $\mathcal{H}$: a linear space of functions $X(\omega)$, called random variables, s.t.
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- $(\Omega, \mathcal{H}, \mathbb{E})$, $\mathcal{H}$: a linear space of functions $X(\omega)$, called random variables, s.t.

\[ X_1(\omega), \cdots, X_d(\omega) \in \mathcal{H} \implies \varphi(X_1(\omega), \cdots, X_d(\omega)) \in \mathcal{H}, \]

for all Lipschitz functions $\varphi(x_1, \cdots, x_d)$
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Nonlinear expectation \( \mathbb{E}: \mathcal{H} \mapsto \mathbb{R} \)

\( \mathbb{E} \) is a nonlinear functional

- **Monotonicity:** \( \mathbb{E}[X] \geq \mathbb{E}[Y] \) if \( X \geq Y \)
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Covering the risk of probability uncertainty by sublinear expectation

- Very useful parameterized sublinear models
Covering the risk of probability uncertainty by **sublinear expectation**

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- Sublinear is the necessary to cover the uncertainty of probabilities
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Expectation nonlinearity v.s.
Probability & distribution uncertainty

\[ \mathbb{E}[X] = \max_{\theta \in \Theta} E_\theta[X] = \max_{\theta \in \Theta} \int_{\Omega} X dP_\theta, \]
Expectation nonlinearity v.s. 
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\( \{P_\theta\}_{\theta \in \Theta} \): probability model uncertainty
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\{P_{\theta}\}_{\theta \in \Theta}: probability model uncertainty

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\[ \mathbb{F}[\varphi] = \sup_{\theta \in \Theta} F_{\theta}[\varphi] = \sup_{\theta \in \Theta} \int_{\Omega} \varphi(X) dP_{\theta} \]
Uncertainty version of I.I.D. of two random variables $X$ and $Y$

**Definition**

- The **distribution uncertainty** of a random $X$ equals to (resp. stronger than) that of $Y$, if for any test function $\varphi(x)$,

$$
E[\varphi(X)] = E[\varphi(Y)],
$$

denoted by $X \overset{d}{=} Y$ (resp. $E[\varphi(X)] \geq E[\varphi(Y)]$, denoted by $X \overset{d}{\geq} Y$).

$Y$ is independent of $X$ if for any test function $\varphi(x, y)$,

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E[\varphi(X, Y)] = E[E[\varphi(x, Y)]_{x = X}],
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Uncertainty version of I.I.D. of two random variables $X$ and $Y$

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- $Y$ is **independent** of $X$ if for any test function $\varphi(x, y)$,

  $$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=x}].$$
Uncertainty version of I.I.D. time sequence $\{X_i\}_{i=1}^{\infty}$

$$\mathbb{E}[\varphi(X_i)] = \mathbb{E}[\varphi(X_1)], \quad X_{i+1} \text{ is independent of } (X_1, \cdots, X_i), \quad \text{for all } i.$$ 

Remark.

$\{X_i\}_{i=1}^{\infty}$ is an IID $\implies \{\varphi(X_i)\}_{i=1}^{\infty}$ is also IID $\forall \varphi$
At time $t = 1$, we randomly choose a ball from an urn containing Black and White balls ($W_1 + B_1 = 100$)

But we know only $W_1 \in [40, 60]$

$$X_1(\omega) = 1\{W_1 = \text{true}\} - 1\{B_1 = \text{true}\}$$
Repeat this game at $t = 1, 2, 3, \cdots$, 

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Repeat this game at \( t = 1, 2, 3, \ldots \),

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- Each time $i$, $W_i$ is changed within $W_i \in [40, 60]$
- We get a sequence of random variables $\{X_i(\omega)\}_{i=1}^{\infty}$;
- This is a typical case of our daily uncertainty: environment changes all the time
- The output $\{X_i\}_{i=1}^{\infty}$ is IID:
  - $X_1, X_2, \cdots$ are identically distributed (same distribution uncertainty)
  - $X_{i+1}$ is independent of $\{X_i\}_{i=1}^{n}$.
- $S_n = \sum_{i=1}^{n} X_i$: a nonlinear Bernoulli random walk.
- $\{\varphi(X_i)\}_{i=1}^{\infty}$ is also IID, for any given function $\varphi(x)$. 
A general random generator of nonlinear IID sequence

\[ \{X_i\}_{i=1}^{\infty} \]

- The urn \( \Rightarrow \) A generator of random vectors \( X_i \), at time \( t = i \),

\[ X_i(\omega) \overset{d}{=} F_\theta, \quad i = 1, 2, 3, \cdots \]

We can observe the output \( X_i \) at \( t = i \) with

\[ \mathcal{L}(X_i) \in \{ F_\theta \}_{\theta \in \Theta} \]
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- \( X_{i+1} \stackrel{d}{=} X_i \), \( X_{i+1} \) is independent of \( (X_1, X_2, \cdots, X_i) \)
An important special case: I.I.D. maximal distributed sequence $\{X_i\}_{i=1}^\infty$

- $\{F_\theta\}_{\theta \in \Theta} = M_{[\mu, \bar{\mu}]}$: all prob. distributions concentrated on $[\mu, \bar{\mu}]$.

- Many cases, we have

$$\mathcal{L}(X_i) \in \{F_\theta\}_{\theta \in \Theta_i}, \quad \Theta_i \subset \Theta$$

But we use $X_i \stackrel{d}{=} \{F_\theta\}_{\theta \in \Theta}$ to robustly cover the uncertainty.
Important advantage of nonlinear expectation space

- **Example**: Maximally-distributed i.i.d sequence \( \{X_i\}_{i=1}^{\infty} \): covers all possible distributions of the sequence satisfying

\[
\underline{\mu} \leq X_i(\omega) \leq \bar{\mu}, \quad i = 1, 2, \ldots \quad \mathbb{E}[\varphi(X_i)] = \max_{\nu \in [\underline{\mu}, \bar{\mu}]} \varphi(\nu)
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- Thus, principally, all random sequences can be treated as an nonlinear IID random sequence.
- How to narrow down \( \bar{\mu} - \underline{\mu} \) through real data is our important task.
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- Thus, principally, all random sequences can be treated as an nonlinear IID random sequence.
- How to narrow down \( \bar{\mu} - \underline{\mu} \) through real data is our important task.
- Challenging objective: to establish a systematic framework \( (\Omega, \mathcal{H}, \mathbb{E}) \) compatible with \( (\Omega, \mathcal{F}, P) \).
- Important: hidden behind are PDEs (linear/nonlinear heat equation)!

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Two fundamentally important nonlinear distributions

- $Y \overset{d}{=} M_{[\mu, \mu]}$ is defined by $aY + b\bar{Y} \overset{d}{=} (a + b)Y$;
- $X \overset{d}{=} N(0, [\sigma^2, \bar{\sigma}^2])$ is defined by $aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X$;
The universality of the IID assumption under sublinear expectation

- For a bounded random sequence \( \{X_i\}_{i=1}^{\infty} \) one can always enlarge the probability uncertainty so that \( \{X_i\}_{i=1}^{\infty} \) is an I.I.D. sequence under the enforced sublinear expectation \( \mathbb{E} \).

**Lemma**

Assume that a random sequence \( \{X_i\}_{i=1}^{\infty} \) is bounded by \( \underline{\mu} \leq X_i \leq \bar{\mu} \). Then we have \( X_i \overset{d}{\leq} \bar{X}_i \) and \( \{X_i\}_{i=1}^{\infty} \overset{d}{\leq} \{\bar{X}_i\}_{i=1}^{\infty} \), where \( \{\bar{X}_i\}_{i=1}^{\infty} \) is an IID sequence with \( \bar{X}_i \overset{d}{=} M_{[\underline{\mu}, \bar{\mu}]} \).
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**Lemma**

Assume that a random sequence \( \{X_i\}_{i=1}^{\infty} \) is bounded by \( \underline{\mu} \leq X_i \leq \overline{\mu} \). Then we have \( X_i \sim \overline{X}_i \) and \( \{X_i\}_{i=1}^{\infty} \sim \{\overline{X}_i\}_{i=1}^{\infty} \), where \( \{\overline{X}_i\}_{i=1}^{\infty} \) is an IID sequence with \( \overline{X}_i \sim M[\underline{\mu}, \overline{\mu}] \).

Thus one can robustly enforce \( \mathbb{E}[\cdot] \) and assume that \( \{X_i\}_{i=1}^{\infty} \) is IID under \( \mathbb{E}[\cdot] \).
The universality of the IID assumption under sublinear expectation

- If a random sequence \( \{ X_i \}_{i=1}^{\infty} \) is not bounded, we can set

\[
Y_i^{(j)} = \frac{2}{\pi} \arctan(X_i^{(j)}), \quad j = 1, 2, \ldots, d.
\]

Then \( \{ Y_i \}_{i=1}^{\infty} \) can be robustly assumed to be IID.

- On the other hand, the ‘IID’ assumption is very flexible and can cover many important cases.

- **Example** If \( \{ X_i \}_{i=1}^{\infty} \) is i.i.d. and the distribution of \( X_1 \) is linear, then it becomes to IID sequence in the classical case.
Theorem (Peng2007)

Let \( \{Y_i\}_{i=1}^{\infty} \) be IID sequence. Assume
\[
\lim_{c \to \infty} \mathbb{E}[|Y_1 - c|] = 0.
\]

Then, for each \( \phi \in C^b(\mathbb{R}) \),
\[
\lim_{n \to \infty} \mathbb{E}[\phi(Y_1 + \cdots + Y_n)] = \mathbb{E}[\phi(Y_1) = \max_{v \in [\mu, \mu]} \phi(v)].
\]

where \( \mu = \mathbb{E}[Y_1] \), \( \mu = -\mathbb{E}[-Y_1] \).

\( Y_d = \) Maximal distribution.

\( u(x, t) = \) solves the 1st order PDE.
Theorem (Peng2007)

Let \( \{Y_i\}_{i=1}^{\infty} \) be IID sequence. Assume \( \lim_{c \to \infty} E[(|Y_1| - c)^+] = 0. \)

Then, for each \( \varphi \in C_b(\mathbb{R}) \),

\[
\lim_{n \to \infty} E\left[ \varphi\left( \frac{Y_1 + \cdots + Y_n}{n} \right) \right] = E\left[ \varphi(Y) \right] = \max_{\nu \in [\mu, \bar{\mu}]} \varphi(\nu).
\]

where \( \bar{\mu} = E[Y_1] \), \( \mu = -E[-Y_1] \).
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\[ Y \overset{d}{=} M_{[\bar{\mu}, \underline{\mu}]} : \text{Maximal distribution.} \]

\[ u(x, t) := \mathbb{E}[\varphi(x + (1 - t)Y)] \] solves the 1st order PDE
Nonlinear Central limit theorem

Theorem (Peng2008-2010)

Let \( \{X_i\}_{i=1}^{\infty} \) be IID sequence. Assume furthermore

\[
\mathbb{E}[|X_1|^{2+\epsilon}] < \infty \quad \mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0
\]

Then, for each \( \varphi \in C_b(\mathbb{R}) \),

\[
\lim_{n \to \infty} \mathbb{E}[\varphi(\frac{X_1 + \cdots + X_n}{\sqrt{n}})] = u^{\varphi}(0,0) = \mathcal{N}_{G}(\varphi).
\]

\[
\partial_t u^{\varphi} + G(\partial_{xx}^{2} u^{\varphi}) = 0, \quad t \in [0,1], \quad u^{\varphi}(1,x) = \varphi(x),
\]

\[
G(a) := \frac{1}{2}[\bar{\sigma}^2 a^+ - \bar{\sigma}^2 a^-]
\]

\[
\bar{\sigma}^2 := \mathbb{E}[X_1^2], \quad \sigma^2 := -\mathbb{E}[-X_1^2] > 0,
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\]
Sketch of the proof: \( u \in C^{1+\frac{\alpha}{2}, 2+\alpha} [Krylov] \)

\[
\lim_{n \to \infty} \mathbb{E}[\varphi(S^n)] = u^\varphi(0, 0),
\]

\[
S^n_k := \frac{1}{\sqrt{n}}(X_1 + \cdots + X_k), \quad k = 1, \ldots, n.
\]

\[
u(S^n, 1) - u(0, 0) = \sum_{k=0}^{n-1} \left[ u(S^n_{k+1}, \frac{k+1}{n}) - u(S^n_k, \frac{k}{n}) \right]
\]
Sketch of the proof: \( u \in C^{1+\frac{\alpha}{2},2+\alpha} \) [Krylov]

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\lim_{n \to \infty} \mathbb{E}[\varphi(S_n^k)] = u^\varphi(0,0),
\]

\[
S_n^k := \frac{1}{\sqrt{n}}(X_1 + \cdots + X_k), \quad k = 1, \ldots, n.
\]

\[
u(S_n^k, 1) - u(0,0) = \sum_{k=0}^{n-1} [u(S_{k+1}^n, \frac{k+1}{n}) - u(S_k^n, \frac{k}{n})]
\]

\[
\mathbb{E}[u(S_{k+1}^n, \frac{k+1}{n}) - u(S_k^n, \frac{k}{n})] = \mathbb{E}[u(S_k^n + \frac{1}{\sqrt{n}}X_{k+1}, \frac{k}{n} + \frac{1}{n}) - u(S_k^n, \frac{k}{n})]
\]

\[
= \mathbb{E}\left[\frac{1}{n}\partial_t u(x, t) + \frac{X_{k+1}}{\sqrt{n}}\partial_x u(x, t) + \frac{X_{k+1}^2}{2n} \partial_{xx} u(x, t)\right] + o\left(\frac{1}{n}\right)
\]

\[
= 0 + o\left(\frac{1}{n}\right), \quad (x, t) = (S_k^n, \frac{k}{n})
\]
Convergence rate of LLN

Theorem (Fang-Peng-Shao-Song(2017))

We have

\[
\mathbb{E}[d^2_{\mu, \bar{\mu}}(X_n)] = \mathbb{E}[((\bar{X}_n - \bar{\mu})^+)^2 + ((\bar{X}_n - \bar{\mu})^-)^2] \leq \frac{2[\bar{\sigma}^2 + (\bar{\mu} - \mu)^2]}{n},
\]

(1)

where

\[\bar{\mu} = \mathbb{E}[X_1], \quad \mu = -\mathbb{E}[-X_1].\]

and

\[\bar{\sigma}^2 := \sup_{\theta \in \Theta} E_{P_\theta}[(X_1 - E_{P_\theta}[X_1])^2].\]
Remark. ([Fang-P.-Shao, Song 2017])

If $\mu > \mu$ then the convergence of $\frac{1}{n}(X_1 + \cdots + X_n)$ to cannot be a strong one!

Stein equation: the key tool of Stein method. Song (2017) had found the corresponding "Stein equation", and provided Nonlinear Stein method.

the rate of convergence is:

\[
\sup_{|\phi|_{\text{Lip}} \leq 1} \left| \hat{E}\left[\phi\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}}\right)\right] - \mathcal{N}_G(\phi) \right| \lesssim \frac{1}{n^\alpha},
\]

where \(\alpha \in (0, 1)\) depends only on \(-\hat{E}[-X_1^2]\) and \(\hat{E}[X_1^2]\).
Converges rate of nonlinear CLT

Theorem (Krylov 2018)

Assume that $|\phi(x) - \phi(y)| \leq |x - y|^\beta$, \( M_\beta := \sup_{\xi \in \Theta} E(|\xi|^{2+\beta}) < \infty \).

Then

$$\left| E[\phi\left(\frac{1}{n}(Y_1 + \cdots + Y_n)\right)] - E[\phi(Y)] \right| \leq Nn^{-\beta^2/(4+2\beta)}$$

where \( N \) depends only on \( M_\beta \) and \( \sigma^2 \).

- [Song] (2017) Normal approximation by Stein’s method under sublinear expectations”,
Related works

Nonlinear normal distributions

**Definition**

A random variable $X$ in $(\Omega, \mathcal{H}, \mathbb{E})$ is normal if the function

$$u(t, x) := \mathbb{E}[\varphi(x + \sqrt{1 - tX})]$$

is the solution of the nonlinear PDE

$$\partial_t u + G(\partial_{xx} u) = 0, \quad u(1, x) = \varphi(x).$$
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where

- $G(a) = \frac{1}{2} [\bar{\sigma}^2 a^+ - \sigma^2 a^-]$  
  $\bar{\sigma}^2 = \mathbb{E}[X^2], \quad \sigma^2 = -\mathbb{E}[-X^2]$
Classical ‘Monté-Carlo’ approach for estimating $\hat{E}[\varphi(X)]$ through data

- **Key point:** How to obtain $\hat{E}[\varphi(X)]$ through its sample $\{x_i\}_{i=1}^N$?
- In many practice cases: we care about $\hat{E}[\varphi(X)]$ with a specific function $\varphi(x)$:
  a consumption utility function, a contract, a cost function ....
- In a classical probability space $(\Omega, \mathcal{F}, P)$, we can apply LLN to calculate

  $$E[\varphi(X)] \sim \mathbb{M}[\varphi(X)] := \frac{1}{N} \sum_{i=1}^{N} \varphi(x_i)$$

where $\{x_i\}_{i=1}^N$ is an IID sample of $X$. 
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where $\{x_i\}_{i=1}^N$ is an IID sample of $X$.
- But: Is $\{x_i\}_{i=1}^N$ a classical IID?
$\varphi$-max-mean algorithm: the data-based distribution of $X$

- Let $(\Omega, \mathcal{H}, \hat{E})$ be a sublinear expectation space and 
  \[
  \{x_i\}_{i=1}^{n \times m} : \text{ IID sample of a random vector } X
  \]

- The max-mean algorithm to estimate $\hat{E}[\varphi(X)]$:
  \[
  \hat{M}[\varphi] = \max\{ Y^k_n : k = 0, \cdots, m - 1 \},
  \]
  where
  \[
  Y^k_n = \frac{1}{n} \sum_{i=1}^{n} \varphi(x_{nk+i}).
  \]

- By the above LLN, as $n \to \infty$, \( \{ Y^k_n \}_{k=0}^{m-1} \xrightarrow{d} \) an IID \( \{ Y^k \}_{k=0}^{m-1} \),
  \[
  Y^k \xrightarrow{d} M_{[\mu, \overline{\mu}]}, \quad \text{with} \quad \overline{\mu} = \hat{E}[\varphi(X)], \quad \mu = -\hat{E}[-\varphi(X)]
  \]

- But $\max\{ Y^k_n : k = 0, \cdots, m - 1 \}$ provides us the asymptotically optimal unbiased estimate.
\( \text{\textbf{\( \varphi \)-max-mean algorithm of \( X \) from IID sample \( \{x_i\}_{i=1}^{mn} \)}} \)

\[
\max \left\{ \left. \frac{\varphi(X_1) + \cdots + \varphi(X_n)}{n} \right| \gamma_0^n \right\}, \ldots, \left. \frac{\varphi(X_{(m-1)n+1}) + \cdots + \varphi(X_{mn})}{n} \right| \gamma_m^n \right\}
\]

\[
\mathbb{E}[\varphi(X)] \simeq \text{Max-Mean-} [\varphi(\{x_i\})] \\
= \max_{0 \leq k \leq m-1} \frac{\sum_{i=1}^{n} \varphi(x_{kn+i})}{n}
\]
Optimality of the estimate

The optimality of the above estimate is based on the following quite simple, but very fundamental result:

**Theorem (Jin-Peng2016)**

Let $Y^1, \ldots, Y^m$ be IID and maximally distributed:

$$Y^i \overset{d}{=} M_{[\mu, \bar{\mu}]}, \quad i = 1, \ldots, m,$$

where $\mu \leq \bar{\mu}$ is two unknown parameters. Then
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**Theorem (Jin-Peng 2016)**

Let $Y^1, \cdots, Y^m$ be IID and maximally distributed:

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where $\mu \leq \overline{\mu}$ is two unknown parameters. Then

$$\mu \leq \min \{ Y^1(\omega), \cdots, Y^n(\omega) \} \leq \max \{ Y^1(\omega), \cdots, Y^n(\omega) \} \leq \overline{\mu}.$$

Moreover

$$\widehat{\mu}_n = \max \{ Y^1, \cdots, Y^n \},$$

is the maximum unbiased estimate of $\overline{\mu}$.
• Many typical nonlinear distributions

\[ M_{[\mu, \overline{\mu}]}, \quad N(\mu, [\sigma^2, \overline{\sigma}^2]), \quad P_{[\lambda, \overline{\lambda}]} \text{ (Nonlinear Poisson)} \]

• asymptotically unbiased estimates: 

\[ \hat{\sigma}^2 := \min_{1 \leq k \leq m} \sigma_k^2, \quad \overline{\sigma}^2 := \max_{1 \leq k \leq m} \sigma_k^2 \]

where \[ \sigma_k^2 := \frac{1}{n} \sum_{j=1}^{n} (x_{n(k-1)+j} - \mu)^2. \]
Maximal and Normal distributions and Nonlinear PDEs

- $Y \overset{d}{=} M_{[\mu, \mu]}$ defined by $aY + b\bar{Y} \overset{d}{=} (a + b)Y$; is directly related to the 1st order PDE

$$\partial_t u^\varphi + g(\partial_x u^\varphi) = 0,$$

$$u^\varphi(x, 1) = \varphi(x).$$

for $u^\varphi(t, x)$ on $t \in [0, 1]$, $x \in \mathbb{R}^d$.

- Through

$$F_g[\varphi] := u^\varphi(0, 0) = \mathbb{E}[\varphi(Y)].$$
Maximal and Normal distributions and Nonlinear PDEs

- $X \overset{d}{=} N(0, [\sigma^2, \bar{\sigma}^2])$ defined by $aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X$ is calculated by the 2nd order parabolic PDEs

\[
\partial_t \nu^\varphi + G(\partial_{xx}^2 \nu^\varphi) = 0,
\]
\[
\nu^\varphi(x, 1) = \varphi(x).
\]

Through:

\[
\mathcal{F}_G[\varphi] := \nu^\varphi(0, 0) = \mathbb{E}[\varphi(X)]
\]
\[
\begin{cases}
\partial_t u(t, x) + g(\partial_x u) = 0 \\
u(T, x) = \varphi(x)
\end{cases}
\]
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u(T,x) = \varphi(x)
\end{cases}
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\[g(a) : \bar{\mu}a\]

\[G(a) : \frac{\sigma^2}{2} \frac{a^2}{2}\]
Papers on statistics and data-analysis under uncertainty


Works on limit theory with nonlinear expectations

Nonlinear Brownian Motion: (Continuous time i.i.d)

Definition.

\( B \) is called a \( G \)-Brownian motion if:

\[
\text{For each } t_1 \leq \cdots \leq t_n, \quad B_{t_n} - B_{t_{n-1}} \text{ is indep. of } (B_{t_1}, \cdots, B_{t_{n-1}}).
\]

\[
B_{t+d} = B_s + (t-s), \quad \text{for all } s, t \geq 0.
\]

\[
\E[|B_t|^3] = o(t).
\]

Theorem.

If \((B_t)_{t \geq 0}\) is a \( G \)-Brownian motion and \( \E[B_t] = \E[-B_t] \equiv 0 \) then:

\[
B_{t+s} - B_s \overset{d}{=} \mathcal{N}(0, \sigma^2 t), \quad \forall s, t \geq 0.
\]
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Theorem.

If $(B_t)_{t \geq 0}$ is a $G$–Brownian motion and $\mathbb{E}[B_t] = \mathbb{E}[-B_t] \equiv 0$ then:

$B_{t+s} - B_s \overset{d}{=} N(0, [\sigma^2 t, \bar{\sigma}^2 t]), \ \forall \ s, t \geq 0$
Real case study: from VaR to GVaR
Problem challenged by CFFEX
(China Financial Future Exchange)

\[ \text{VaR}(X) = -\inf \{ x \mid P(X \leq x) > \alpha \} = -\inf \{ x \mid F(x) > \alpha \} \]

Can we use \( G \)-normal distribution in the place of a linear distribution \( F \)?

Nonlinear normally distributed VaR — G-VaR:
\[ X_{t+1} = N_G. \{X_t\} \text{ of daily return data of CSI300, April 13, 2010–April 16, 2015; } \{X_t\} \text{: S&P 500 daily returns from 04/03/2010 to 09/12/2014,} \]
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\[ \text{VaR}^F_\alpha(X) = - \inf \{ x \mid P(X \leq x) > \alpha \} \]
\[ = - \inf \{ x \mid F(x) > \alpha \}, \]
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- \( \{X_t\} \): of daily return data of CSI300, April 13, 2010- April 16, 2015;
- \( \{X_t\} \): S&P 500 daily returns from 04/03/2010 to 09/12/2014,
Empirical test of robust VaR

\[ \alpha(X) = -\inf_{x \in \mathbb{R}} \{ F_G(x) > \alpha \} \]

We have

\[ F_G(x) = u(t, x) \mid t = 0, \]

where \( u \) is the solution of the PDE

\[ \partial_t u + G(\partial_{xx} u) = 0, \]

(2)

with the Cauchy condition

\[ u(1, x) = 1 \left[ 0, \infty \right)(x). \]

(3)

\[ F_G(x) \] has the explicit expression:

\[ F_G(x) = \hat{x} - \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left( \sigma + \sigma \right)^2 \left[ \exp \left( \frac{-y^2}{2\sigma^2} \right) \right]_{y \leq 0} + \exp \left( \frac{-y^2}{2\sigma^2} \right)_{y > 0} \, dy. \]

(4)
Empirical test of robust VaR

\[ \text{GVaR}_\alpha(X) := - \inf \{ x \in \mathbb{R} : F_G(x) > \alpha \} . \]

We have

\[ F_G(x) = u(t, x) \big|_{t=0}, \]

\[ F^*_G(x) \] has the explicit expression:

\[ F^*_G(x) = \frac{x - \infty}{\sqrt{2\sqrt{\pi}}} (\sigma + \sigma) \] with

\[ \int_{-\infty}^{0} \exp \left( -\frac{y^2}{2\sigma^2} \right) dy + \int_{0}^{\infty} \exp \left( -\frac{y^2}{2\sigma^2} \right) dy. \]
Empirical test of robust VaR

\[ \text{GVaR}_\alpha(X) := - \inf \{ x \in \mathbb{R} : F_G(x) > \alpha \} \].

We have

\[ F_G(x) = u(t, x)|_{t=0}, \]

\( u \) is the solution of the PDE

\[ \partial_t u + G(\partial_{xx}^2 u) = 0, \quad (2) \]

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\[ u(1, x) = 1_{[0, \infty)}(x). \quad (3) \]
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$F_G$ has the explicit expression:

$$F_G(x) = \int_{-\infty}^{x} \frac{\sqrt{2}}{\sqrt{\pi(\sigma^2 + \sigma^2)}} \left[ \exp\left(\frac{-y^2}{2\sigma^2}\right) 1_{y \leq 0} + \exp\left(\frac{-y^2}{2\sigma^2}\right) 1_{y > 0} \right] dy.$$  \hspace{1cm} (4)
$X_{t+1}$ is assumed to be $G$-normally distributed:

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• $X_{t+1}$ is assumed to be $G$-normally distributed:

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• For each $\bar{t}$, use the passed 1 year data $\{X_{\bar{t}-s}\}_{0 \leq s \leq l-1}$ to estimate two parameters $\sigma^2_{\bar{t}}$ and $\bar{\sigma}^2_{\bar{t}}$ at the day $\bar{t}$:
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For each \( \bar{t} \), use the passed 1 year data \( \{X_{\bar{t}-s}\}_{0 \leq s \leq l-1} \) to estimate two parameters \( \sigma^2_{\bar{t}} \) and \( \bar{\sigma}^2_{\bar{t}} \) at the day \( \bar{t} \):

- Fix a window width \( w = 100 \) use the moving window

\[
\sigma^2_{\bar{t},w} := \sigma^2(X_{\bar{t}-w+1}, \cdots, X_{\bar{t}}).
\]
Then get the upper and low data variances:

\[ \bar{\sigma}^2_t = \max\{\sigma^2_{t,20}, \sigma^2_{t-s,w}; \ s \in [0, \cdots, l - w]\}, \]
\[ \underline{\sigma}^2_t = \min\{\sigma^2_{t,20}, \sigma^2_{t-s,w}; \ s \in [0, \cdots, l - w]\}. \]
Then get the upper and low data variances:

\[ \sigma^2_t = \max\{\sigma^2_{t,20}, \sigma^2_{t-s,w}; s \in [0, \cdots, l-w]\}, \]

\[ \sigma^2_{-t} = \min\{\sigma^2_{t,20}, \sigma^2_{t-s,w}; s \in [0, \cdots, l-w]\}. \]

\[ \sigma^2_{t,w}(X_{t-w+1}, \cdots, X_t) \] is the std of \((X_{t-w+1}, \cdots, X_t)\).
Then get the upper and low data variances:

\[
\sigma^2_{\bar{t}} = \max\{\sigma^2_{t,20}, \sigma^2_{\bar{t}-s,w}; \ s \in [0, \cdots, l - w]\},
\]
\[
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\sigma^2_{\bar{t},w}(X_{\bar{t}-w+1}, \cdots, X_{\bar{t}}) \text{ is the std of } (X_{\bar{t}-w+1}, \cdots, X_{\bar{t}}).
\]

\[
\text{GVar}_{\alpha,\hat{t}}(X_{\hat{t}+1}) = -\max\{x: \ F_{G_{\hat{t}}}(x) \leq \alpha\}.
\]
Then get the upper and low data variances:

\[ \sigma^2_{\bar{t}} = \max\{\sigma^2_{\bar{t},20}, \sigma^2_{\bar{t}-s,w}; \ s \in [0, \cdots, l-w]\}, \]
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\[ \text{GVaR}_{\alpha,\bar{t}}(X_{\bar{t}+1}) = -\max\{x : F_{G_{\bar{t}}}(x) \leq \alpha\}. \]

\[ F_G(x) = \int_{-\infty}^{x} \frac{\sqrt{2}}{\sqrt{\pi(\bar{\sigma} + \sigma)^2}} \left[ \exp\left(\frac{-y^2}{2\sigma^2}\right)1_{y \leq 0} + \exp\left(\frac{-y^2}{2\sigma^2}\right)1_{y > 0} \right] dy. \]

(5)
Comparison:
Nonlinear expectation theory, especially sub linear expectation theory is an important tool to "hedge" the probability distribution uncertainty.

Maximal distribution and nonlinear normal distribution is fundamental, and can quantitatively cover most real world cases.
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Maximal distribution and nonlinear normal distribution is fundamental, and can quantitatively cover most real world cases.

The max-mean algorithm, based on the nonlinear LLN gives us the asymptotical optimal estimate of the nonlinear mean $\mathbb{E}[\varphi(X)]$ of $X$ through its real data sample $\{x_i\}$ is very robust.

This algorithm provide us automatically the degree of uncertainty, through the degree of its nonlinearity.
According to our new law of large number, maximal distribution $M_{[\mu, \bar{\mu}]}$ is the other typical case which was often treated as a constant.
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Need a deep collaboration of experts from probability, functional analysis, PDE and stochastic PDE, scientific computing, especially with experts of economics and statistics to develop this new, deep directions.
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Need a deep collaboration of experts from probability, functional analysis, PDE and stochastic PDE, scientific computing, especially with experts of economics and statistics to develop this new, deep directions.

Combining with machine learning, to get more robust and deeper understanding the information we can obtain through a real and dynamical sample \( \{x_i\} \).
A Nicole

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